Mathematical Analysis of Random Noise

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PART III

STATISTICAL PROPERTIES OF RANDOM NOISE CURRENTS

3.0 Introduction

In this section we use the representations of the noise currents given in section 2.8 to derive some statistical properties of $I(t)$. The first six sections are concerned with the probability distribution of $I(t)$ and of its zeros and maxima. Sections 3.7 and 3.8 are concerned with the statistical properties of the envelope of $I(t)$. Fluctuations of integrals involving $I^2(t)$ are discussed in section 3.9. The probability distribution of a sine wave plus a noise current is given in 3.10 and in 3.11 an alternative method of deriving the results of Part III is mentioned. Prof. Uhlenbeck has pointed out that much of the material in this Part is closely connected with the theory of Markoff processes. Also S. Chandrasekhar has written a review of a class of physical problems which is related, in a general way, to the present subject.[22]

3.1 The Distribution of the Noise Current[23]

In section 1.4 it has been shown that the distribution of a shot effect current approaches a normal law as the expected number of events per second, ν, increases without limit.

In line with the spirit of this Part, Part III, we shall use the representation

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \qquad (2.8\text{-}1)$$

to show that $I(t)$ is distributed according to a normal law. This is obtained at once when the procedure outlined in section 2.8 is followed. Since a_n and b_n are distributed normally, so are $a_n \cos \omega_n t$ and $b_n \sin \omega_n t$ when t is regarded as fixed. $I(t)$ is thus the sum of $2N$ independent normal variates and consequently is itself distributed normally.

[22] Stochastic Problems in Physics and Astronomy, *Rev. of Mod. Phys.*, Vol. 15, pp. 1–89 (1943).

[23] An interesting discussion of this subject by V. D. Landon and K. A. Norton is given in the *I.R.E. Proc.*, 30 (Sept. 1942) pp. 425–429.

The average value of $I(t)$ as given by (2.8–1) is zero since $\bar{a}_n = \bar{b}_n = 0$:

$$\bar{I}(t) = 0 \tag{3.1–1}$$

The mean square value of $I(t)$ is

$$\begin{aligned} \overline{I^2(t)} &= \sum_{n=1}^{N} (\overline{a_n^2} \cos^2 \omega_n t + \overline{b_n^2} \sin^2 \omega_n t) \\ &= \sum_{n=1}^{N} w(f_n)\Delta f \qquad (3.1–2) \\ &\to \int_0^\infty w(f)\, df = \psi(0) \equiv \psi_0 \end{aligned}$$

In writing down (3.1–2) we have made use of the fact that all the a's and b's are independent and consequently the average of any cross product is zero. We have also made use of

$$\overline{a_n^2} = \overline{b_n^2} = w(f_n)\Delta f, \qquad f_n = n\Delta f, \qquad \omega_n = 2\pi f_n$$

which were given in 2.8. $\psi(\tau)$ is the correlation function of $I(t)$ and is related to $w(f)$ by

$$\psi_\tau \equiv \psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau \, df \tag{2.1–6}$$

as is explained in section 2.1. In this part we shall write the argument of $\psi(\tau)$ as a subscript in order to save space.

Since we know that $I(t)$ is normal and since we also know that its average is zero and its mean square value is ψ_0, we may write down its probability density function at once. Thus, the probability of $I(t)$ being in the range $I, I + dI$ is

$$\frac{dI}{\sqrt{2\pi\psi_0}} e^{-I^2/2\psi_0} \tag{3.1–3}$$

This is the probability of finding the current between I and $I + dI$ at a time selected at random. Another way of saying the same thing is to state that (3.1–3) is the fraction of time the current spends in the range $I, I + dI$.

In many cases it is more convenient to use the representation (2.8–6)

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \varphi_n), \qquad c_n^2 = 2w(f_n)\Delta f \tag{2.8–6}$$

in which $\varphi_1, \cdots \varphi_n$ are independent random phase angles. In order to deduce the normal distribution from this representation we first observe

that (2.8–6) expresses $I(t)$ as the sum of a large number of independent random variables

$$I(t) = x_1 + x_2 + \cdots + x_N$$
$$x_n = c_n \cos(\omega_n t - \varphi_n)$$

and hence that as $N \to \infty$ $I(t)$ becomes distributed according to a normal law. In order to make the limiting process definite we first choose N and Δf such that $N\Delta f = F$ where

$$\int_F^\infty w(f)\, df < \epsilon \int_0^\infty w(f)\, df$$

where ϵ is some arbitrarily chosen small positive quantity. We now let $N \to \infty$ and $\Delta f \to 0$ in such a way that $N\Delta f$ remains equal to F. Then

$$\begin{aligned} A = \overline{x_1^2} + \overline{x_2^2} + \cdots + \overline{x_N^2} &= \sum_1^N 2w(f_n)\Delta f\, \overline{\cos^2(\omega_n t - \varphi_n)} \\ &= \sum_1^N w(f_n)\Delta f \to \int_0^F w(f)\, df \end{aligned} \tag{3.1–4}$$

$$\begin{aligned} B = \overline{|x_1|^3} + \cdots + \overline{|x_N|^3} &= \sum_1^N (2w(f_n)\Delta f)^{3/2}\, \overline{|\cos(\omega_n t - \varphi_n)|^3} \\ &< 4(\Delta f)^{1/2} \int_0^F [w(f)]^{3/2}\, df \end{aligned}$$

where the bars denote averages with respect to the φ's, t being held constant. If we assume that the integrals are proper, the ratio $BA^{-3/2} \to 0$ as $N \to \infty$, and consequently the central limit theorem* may be used if $w(f) = 0$ for $f > F$. Since we may make F as large as we please by choosing ϵ small enough, we may cover as large a frequency range as we wish. For this reason we write ∞ in place of F.

Now that the central limit theorem has told us that the distribution of $I(t)$, as given by (2.8–6), approaches a normal law, there remains only the problem of finding the average and the standard deviation:

$$\begin{aligned} \overline{I(t)} &= \sum_1^N c_n\, \overline{\cos(\omega_n t - \varphi_n)} = 0 \\ \overline{I^2(t)} &= \sum_1^N c_n^2\, \overline{\cos^2(\omega_n t - \varphi_n)} \\ &\to \int_0^\infty w(f)\, df = \psi_0 \end{aligned} \tag{3.1–5}$$

* Section 2.10.

This gives the probability density (3.1–3). Hence the two representations lead to the same result in this case. Evidently, they will continue to lead to identical results as long as the central limit theorem may be used. In the future use of the representation (2.8–6) we shall merely assume that the central limit theorem may be applied to show that a normal distribution is approached. We shall omit the work corresponding to equations (3.1–4).

The characteristic function for the distribution of $I(t)$ is

$$\text{ave. } e^{iuI(t)} = \exp -\frac{\psi_0}{2} u^2 \tag{3.1–6}$$

3.2 The Distribution of $I(t)$ and $I(t + \tau)$

We require the two dimensional distribution in which the first variable is the noise current $I(t)$ and the second variable is its value $I(t + \tau)$ at some later time τ. It turns out that this distribution is normal[24], as we might expect from the analogy with section 3.1. The second moments of this distribution are

$$\begin{aligned} \mu_{11} &= \overline{I^2(t)} = \psi_0 = \int_0^\infty w(f)\, df \\ \mu_{22} &= \psi_0 \\ \mu_{12} &= \overline{I(t)I(t+\tau)} \\ &= \psi_\tau \end{aligned} \tag{3.2–1}$$

The expression for μ_{12} is in line with our definition (2.1–4) for the correlation function:

$$\psi_\tau \equiv \psi(\tau) = \operatorname*{Limit}_{T\to\infty} \frac{1}{T} \int_0^T I(t)I(t+\tau)\, dt \tag{2.1–4}$$

In order to get the distribution from the representation (2.8–6) we write

$$I_1 = I(t) = \sum_1^N c_n \cos(\omega_n t - \varphi_n)$$

$$I_2 = I(t+\tau) = \sum_1^N c_n \cos(\omega_n t - \varphi_n + \omega_n \tau)$$

[24] It seems that the first person to obtain this distribution in connection with noise was H. Thiede, *Elec. Nachr. Tek.* 13 (1936), 84–95.

From the central limit theorem for two dimensions it follows that I_1 and I_2 are distributed normally. As in (3.1)

$$\mu_{11} = \overline{I_1^2} = \sum_1^N c_n^2 \cdot \tfrac{1}{2} \to \int_0^\infty w(f)\, df = \psi_0$$

$$\mu_{22} = \overline{I_2^2} = \overline{I_1^2} = \psi_0 \tag{3.1–2}$$

$$\mu_{12} = \overline{I_1 I_2} = \sum_1^N c_n^2 \text{ ave. } \{\cos(\omega_n t - \varphi_n)\cos(\omega_n t - \varphi_n + \omega_n \tau)\}$$

Now the quantity within the parenthesis is

$$\cos^2(\omega_n t - \varphi_n)\cos\omega_n\tau - \cos(\omega_n t - \varphi_n)\sin(\omega_n t - \varphi_n)\sin\omega_n\tau$$

and when we take the average with respect to φ_n the second term drops out, giving

$$\mu_{12} = \sum_1^N c_n^2 \cdot \tfrac{1}{2}\cos\omega_n\tau \to \int_0^\infty w(f)\cos 2\pi f\tau\, df = \psi_\tau \tag{3.2–3}$$

where we have used $\omega_n = 2\pi f_n$ and the relation (2.1–6) between $w(f)$ and $\psi(\tau)$.

The probability density function for I_1 and I_2 may be stated. From the discussion of the normal law in 2.9 it is

$$\frac{[\psi_0^2 - \psi_\tau^2]^{-1/2}}{2\pi} \exp\left[\frac{-\psi_0 I_1^2 - \psi_0 I_2^2 + 2\psi_\tau I_1 I_2}{2(\psi_0^2 - \psi_\tau^2)}\right] \tag{3.2–4}$$

For a band pass filter whose range extends from f_a to f_b we have

$$\begin{aligned}\psi_\tau &= \int_{f_a}^{f_b} w_0 \cos 2\pi f\tau\, df \\ &= w_0 \frac{\sin\omega_b\tau - \sin\omega_a\tau}{2\pi\tau} \\ &= \frac{w_0}{\pi\tau}\sin\pi\tau(f_b - f_a)\cos\pi\tau(f_b + f_a) \\ \psi_0 &= w_0(f_b - f_a)\end{aligned} \tag{3.2–5}$$

where w_0 is the constant value of $w(f)$ in the pass band and

$$\begin{aligned}\omega_b &= 2\pi f_b \\ \omega_a &= 2\pi f_a\end{aligned} \tag{3.2–6}$$

According to our formula (3.2–4), I_1 and I_2 are independent when ψ_τ is zero. For the τ's which make ψ_τ zero, a knowledge of I_1 does not add to our knowledge of I_2. For example, suppose we have a narrow filter. Then

$$\psi_\tau = 0 \text{ when } \tau = [2(f_b + f_a)]^{-1}$$

$$\psi_\tau \text{ is nearly } -\psi_0 \text{ when } \tau = [f_b + f_a]^{-1}$$

For the first value of τ, all we know is that I_2 is distributed about zero with $\overline{I_2^2} = \psi_0$. For the second value of τ I_2 is likely to be near $-I_1$. This is in line with the idea that the noise current through a narrow filter behaves like a sine wave of frequency $\frac{1}{2}(f_b + f_a)$ (and, incidentally, whose amplitude fluctuates with an irregular frequency of the order of $\frac{1}{2}(f_b - f_a)$). The first value of τ corresponds to a quarter-period of such a wave and the second value to a half-period. By drawing a sine wave and looking at points separated by quarter and half periods, the reader will see how the ideas agree.

The characteristic function for the distribution of I_1 and I_2 is

$$\text{ave. } e^{iuI_1+ivI_2} = \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \tag{3.2-7}$$

The three dimensional distribution in which

$$I_1 = I(t)$$

$$I_2 = I(t + \tau_1)$$

$$I_3 = I(t + \tau_1 + \tau_2)$$

where τ_1 and τ_2 are given and t is chosen at random is, as we might expect, normal in three dimensions. The moments, from which the distribution may be obtained by the method of Section 2.9, are

$$\mu_{11} = \mu_{22} = \mu_{33} = \psi_0$$

$$\mu_{12} = \psi_{\tau_1}$$

$$\mu_{23} = \psi_{\tau_2}$$

$$\mu_{13} = \psi(\tau_1 + \tau_2) = \psi_{\tau_1+\tau_2}$$

The characteristic function for I_1, I_2, I_3 is

$$\text{ave. } e^{iz_1I_1+iz_2I_2+iz_3I_3}$$

$$= \exp\left[-\frac{\psi_0}{2}(z_1^2 + z_2^2 + z_3^2) - \mu_{12}z_1z_2 - \mu_{23}z_2z_3 - \mu_{13}z_1z_3\right] \tag{3.2-8}$$

3.3 Expected Number of Zeros per Second

We shall use the following result. Let y be given by

$$y = F(a_1, a_2, \cdots a_N; x), \tag{3.3-1}$$

and let the a's be random variables. For a given set of a's, this equation gives a curve of y versus x. Since the a's are random variables we shall call this curve a random curve. Let us select a short interval x_1, $x_1 + dx$,

and then draw a batch of a's. The probability that the curve obtained by putting these a's in (3.3–1) will have a zero in x_1, $x_1 + dx$ is

$$dx \int_{-\infty}^{+\infty} |\eta| \, p(0, \eta; x_1) \, d\eta \tag{3.3–2}$$

and the expected number of zeros in the interval (x_1, x_2) is

$$\int_{x_1}^{x_2} dx \int_{-\infty}^{+\infty} |\eta| \, p(0, \eta; x) \, d\eta \tag{3.3–3}$$

In these expressions $p(\xi, \eta; x)$ is the probability density function for the variables

$$\begin{aligned} \xi &= F(a_1, \cdots a_N; x) \\ \eta &= \frac{\partial F}{\partial x} \end{aligned} \tag{3.3–4}$$

Since the a's are random variables so are ξ and η, and their distribution will contain x as a parameter. This is indicated by the notation $p(\xi, \eta; x)$.

These results may be proved in much the same manner as are similar results for the distribution of the maxima of a random curve. This method of proof suffers from the restriction that the a's are required to be bounded.[25] Results equivalent to (3.3–2) and (3.3–3) have been obtained independently by M. Kac.[26] His method of proof has the advantage of not requiring the a's to be bounded.

Here we shall sketch the derivation of a closely related result: The probability that y will pass through zero in x_1, $x_1 + dx$ with positive slope is

$$dx \int_0^{\infty} \eta p(0, \eta; x_1) \, d\eta \tag{3.3–5}$$

We choose dx so small that the portions of all but a negligible fraction of the possible random curves lying in the strip $(x_1, x_1 + dx)$ may be regarded as straight lines. If $y = \xi$ at x_1 and passes through zero for $x_1 < x < x_1 + dx$, its intercept on $y = 0$ is $x_1 - \frac{\xi}{\eta}$ where η is the slope. Thus ξ and η must be of opposite sign and

$$x_1 < x_1 - \frac{\xi}{\eta} < x_1 + dx$$

[25] S. O. Rice, *Amer. Jour. Math.* Vol. 61, pp. 409–416 (1939). However, L. A. MacColl has pointed out to me that a set of sufficient conditions for (3.3–5) to hold is: (a) $p(\xi, \eta; x)$ is continuous with respect to (ξ, η) throughout the $\xi\eta$-plane; and (b) that the integral

$$\int_0^{\infty} p(a\eta, \eta; x_1) \, d\eta$$

converges uniformly with respect to a in some interval $-a_1 \leq a \leq a_2$, where a_1 and a_2 are positive. These conditions are satisfied in all the applications we shall make use of (3.3–5).

[26] M. Kac, *Bull. Amer. Math. Soc.* Vol. 49, pp. 314–320 (1943).

According to the statement of our problem, we are interested only in positive values of η, and we therefore write our inequality as

$$-\eta\, dx < \xi < 0$$

For a given random curve i.e. for a given set of a's ξ and η have the values given by

$$\xi = F(a_1, \cdots a_N ; x_1)$$

$$\eta = \left[\frac{\partial F}{\partial x}\right]_{x=x_1}$$

If these values of ξ and η satisfy our inequality, the curve goes through zero in x_1, $x_1 + dx$. The probability of this happening is[27]

$$\int_0^\infty d\eta \int_{-\eta\, dx}^0 d\xi\, p(\xi, \eta; x_1) = \int_0^\infty [0 - (-\eta\, dx)]p(0, \eta; x_1)\, d\eta$$

where we have made use of the fact that dx is so very small that ξ is effectively zero. The last expression is the same as (3.3–5).

In the same way it may be shown that the probability of y passing through zero in x_1, $x_1 + dx$ with a negative slope is

$$-dx \int_{-\infty}^0 \eta p(0, \eta; x_1)\, d\eta \qquad (3.3\text{–}6)$$

Expression (3.3–2) is obtained by adding (3.3–5) and (3.3–6).

We are now ready to apply our formulas. We let t, $I(t)$ and φ_n play the roles of x, y, and a_n, respectively, and use

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \varphi_n), \qquad c_n^2 = 2w(f)\Delta f \qquad (2.8\text{–}6)$$

[27] MacColl has remarked that the step from the double integral on the left hand side of this equation to the final result (3.3–5) may be made as follows:
It is easily seen that the probability density we are seeking is

$$\left[\frac{d}{d(\Delta x)} \int_0^\infty d\eta \int_{-\eta\Delta x}^0 p(\xi, \eta; x)\, d\xi\right]_{\Delta x=0}$$

Proceeding formally, without regard to conditions validating the analytical operations (for such conditions see the footnote on page 52), we have

$$\frac{d}{d\Delta x} \int_0^\infty d\eta \int_{-\eta\Delta x}^0 p(\xi, \eta; x)\, d\xi = \int_0^\infty \eta p(-\eta\Delta x, \eta; x)\, d\eta$$

and hence the required probability density is

$$\int_0^\infty \eta p(0, \eta; x)\, d\eta$$

The first step is to find the probability density function of the two random variables

$$\xi = \sum_{n=1}^{N} c_n \cos(\omega_n t_1 - \varphi_n)$$
$$\eta = I'(t_1) = -\sum_{n=1}^{N} c_n \omega_n \sin(\omega_n t_1 - \varphi_n) \tag{3.3-7}$$

where the prime denotes differentiation with respect t. From section 2.10

$$\mu_{11} = \overline{\xi^2} = \psi_0$$
$$\mu_{22} = \overline{\eta^2} = \sum_{n=1}^{N} c_n^2 \omega_n^2 \overline{\sin^2(\omega_n t_1 - \varphi_n)}$$
$$= \sum_{n=1}^{N} (2\pi f_n)^2 w(f_n) \Delta f$$
$$\to 4\pi^2 \int_0^\infty f^2 w(f)\, df = -\psi_0''$$
$$\mu_{12} = \overline{\xi\eta} = -\sum_{n=1}^{N} c_n^2 \omega_n \overline{\cos(\omega_n t_1 - \varphi_n) \sin(\omega_n t_1 - \varphi_n)}$$
$$= 0$$

The expression for μ_{22} arises from (2.1–6) by differentiation. In this expression ψ_0'' denotes the second derivative of $\psi(\tau)$ with respect to τ at $\tau = 0$:

$$\psi''(\tau) = -4\pi^2 \int_0^\infty f^2 w(f) \cos 2\pi f\tau\, df \tag{3.3-8}$$

Hence the probability density is

$$p(\xi, \eta; t) = \frac{[-\psi_0\psi_0'']^{-1/2}}{2\pi} \exp\left[-\frac{\xi^2}{2\psi_0} + \frac{\eta^2}{2\psi_0''}\right] \tag{3.3-9}$$

where ψ_0'' is negative. It will be observed that the expression on the right is independent of t. Hence the probability of having a zero in t_1, $t_1 + dt$,

$$dt \int_{-\infty}^{+\infty} |\eta| \frac{[-\psi_0\psi_0'']^{-1/2}}{2\pi} e^{\eta^2/2\psi_0'} d\eta = \frac{dt}{\pi}\left[-\frac{\psi''(0)}{\psi(0)}\right]^{1/2} \tag{3.3-10}$$

which follows from (3.3–3), is independent of t.

The expected number of zeros per second, which may be obtained from (3.3–3) by integrating (3.3–10) over an interval of one second, is

$$\frac{1}{\pi}\left[-\frac{\psi''(0)}{\psi(0)}\right]^{1/2} = 2\left[\frac{\int_0^\infty f^2 w(f)\, df}{\int_0^\infty w(f)\, df}\right]^{1/2} \tag{3.3-11}$$

For an ideal band pass filter whose pass band extends from f_a to f_b the expected number of zeros per second is

$$2\left[\frac{1}{3}\frac{f_b^3 - f_a^3}{f_b - f_a}\right]^{1/2} \tag{3.3-12}$$

When f_a is zero this becomes $1.155\, f_b$ and when f_a is very nearly equal to f_b it approaches $f_b + f_a$.

In a recent paper M. Kac[28] has given a result which, after a slight generalization, leads to

$$e^{-I^2/2\psi_0}\frac{1}{2\pi}\left[-\frac{\psi_0''}{\psi_0}\right]^{1/2} dt \tag{3.3-13}$$

for the probability that the noise current will pass through the value I with positive slope during the interval $t, t + dt$. The expected number of such passages per second is

$$e^{-I^2/2\psi_0} \times [\tfrac{1}{2} \text{ the expected number of zeros per second}] \tag{3.3-14}$$

The expression (3.3–13) may also be derived from analogue of (3.3–5) obtained by replacing the zero in $p(0, \eta; x_1)$ by y.

In some cases the integral

$$\psi_0'' = -4\pi^2 \int_0^\infty f^2 w(f)\, df$$

does not converge.

An example occurs when we apply a broad band noise voltage to a resistance and condenser in series. The power spectrum of the voltage across the condenser is of the form

$$w(f) = \frac{1}{f^2 + a^2} \tag{3.3-15}$$

Although ψ_0'' is infinite, ψ_0 is finite and equal to $\pi/2a$. A straightforward substitution in our formula (3.3–11) gives infinity as the expected number of zeros per second.

Some light is thrown on this breakdown of our formula when we consider a noise current consisting of two bands of noise. One band is confined to relatively low frequencies, and its power spectrum will be denoted by $w_1(f)$. The other band is very narrow and is centered at the relatively high frequency f_2. The complete power spectrum of our noise is then

$$w(f) = w_1(f) + A^2\delta(f - f_2)$$

[28] On the Distribution of Values of Trigonometric Sums with Linearly Independent Frequencies, *Amer. Jour. Math.*, Vol. LXV, pp 609–615, (1943).

where the unit impulse function δ is used to represent the very narrow band. The power spectrum of the narrow band is approximately the same as that of the wave $A\sqrt{2}\cos 2\pi f_2 t$.

The integrals occurring in our formula are

$$\int_0^\infty w(f)\,df = \int_0^\infty w_1(f)\,df + A^2$$
$$= W + A^2$$
$$\int_0^\infty w(f)f^2\,df = \int_0^\infty f^2 w_1(f)\,df + A^2 f_2^2$$
$$= U + A^2 f_2^2$$

We suppose that A and f_2 are such that

$$W \gg A^2$$
$$U \ll A^2 f_2^2 .$$

Then our formula (3.3–11) gives us the expected number of zeros

$$2\frac{Af_2}{W^{1/2}}$$

We may give a qualitative explanation of this formula if we regard our noise current as composed of a small component

$$I_2 = 2^{1/2}A\cos 2\pi f_2 t$$

due to the narrow band superposed on a large, slowly varying component due to the lower band. Since the r.m.s. value of the second component is $W^{1/2}$ we may assign it a representative frequency f_1 and write it approximately as

$$I_1 = (2W)^{1/2}\cos 2\pi f_1 t$$

The zeros of the noise current are clustered around the zeros of the second wave. Near such a zero

$$I_1 = \pm(2W)^{1/2}2\pi f_1 \Delta t$$

where Δt is the distance from the zero. The oscillations of I_1 produce zeros when $|I_1|$ is less than the amplitude of I_2 or when

$$A > W^{1/2}2\pi f_1 |\Delta t|$$

and the interval over which zeros are produced is given by

$$2\Delta t = \frac{AW^{-1/2}}{\pi f_1}$$

The number of zeros is this multiplied by $2f_2$. Since there are $2f_1$ such intervals per second the number of zeros per second is

$$\frac{4}{\pi} AW^{-1/2} f_2$$

This differs from the result given by our formula by a factor of $2/\pi$. This discrepancy is due to our representing the two bands by the sine waves I_1 and I_2.

From this example we obtain the picture that when the integral for ψ_0 converges corresponding to $A \to 0$, while at the same time the integral for ψ_0'' diverges, corresponding to $f_2 \to \infty$ in such a way that $Af_2 \to \infty$, the noise current behaves something like a continuous function which has no derivative. It seems that for physical systems the integrals will always converge since parasitic effects will have the effect of making $w(f)$ tend to zero rapidly enough. The frequency which represents the region where this occurs is of the order of the frequency of the microscopic wiggles.

So far we have been considering the formulas of this section in the most favorable light possible. There are experiments which indicate the possibility of the formulas breaking down in some cases. Prof. Uhlenbeck has pointed out that if a very broad band fluctuation current be forced[29] to flow through a circuit consisting of a condenser, C, in parallel with a series combination of inductance, L, and resistance, R, equation (3.3–11) says that the expected number of zeros per second of the current, I, flowing through R (and L) is independent of R. It is simply $\frac{1}{\pi}(LC)^{-1/2}$. The differential equation for I is the same as that which governs the Brownian motion of a mirror suspended in a gas[30], the gas pressure playing the role of R. Curves are available for this motion and it is seen that their character depends greatly upon the pressure[31]. Unfortunately, it is difficult to tell from the curves whether the expected number of zeros is independent of the pressure. The differences between the curves for various pressures indicates that there may be some dependence*.

3.4 The Distribution of Zeros

The problem of determining the distribution function for the distance between two successive zeros seems to be quite difficult and apparently

[29] For example, by putting the circuit in series with a diode.

[30] This problem in Brownian motion is discussed by G. E. Uhlenbeck and S. Goudsmit, *Phys., Rev.*, 34 (1929), 145–151.

[31] E. Kappler, *Annalen d. Phys.*, 11 (1931) 233–256.

* Since this was written M. Kac and H. Hurwitz have studied the problem of the expected number of zeros using quite a different method of approach which employs the "shot-effect" representation (Sec. 3.11). Their results confirm the correctness of (3.3–11) when the integrals converge. When the integrals diverge the average number of electrons, per sec. producing the shot effect must be considered.

nobody has as yet given a satisfactory solution. Here we shall give some results which are related to the general problem and which give an idea of the form of the distribution for the region of small spacings between the zeros.

We shall show (in the work starting with equation (3.4–12)) that the probability of the noise current, I, passing through zero in the interval $\tau, \tau + d\tau$ with a negative slope, when it is known that I passes through zero at $\tau = 0$ with a positive slope, is

$$\frac{d\tau}{2\pi}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2}\left[\frac{M_{23}}{H}\right](\psi_0^2 - \psi_\tau^2)^{-3/2}[1 + H\cot^{-1}(-H)] \quad (3.4\text{–}1)$$

where M_{22} and M_{23} are the cofactors of $\mu_{22} = -\psi_0''$ and $\mu_{23} = -\psi_\tau''$ in the matrix

$$M = \begin{bmatrix} \psi_0 & 0 & \psi_\tau' & \psi_\tau \\ 0 & -\psi_0'' & -\psi_\tau'' & -\psi_\tau' \\ \psi_\tau' & -\psi_\tau'' & -\psi_0'' & 0 \\ \psi_\tau & -\psi_\tau' & 0 & \psi_0 \end{bmatrix}, \quad (3.4\text{–}2)$$

$$H = M_{23}[M_{22}^2 - M_{23}^2]^{-1/2}.$$

We choose $0 \leq \cot^{-1}(-H) \leq \pi$, the value π being taken at $\tau = 0$, and the value $\pi/2$ being approached as $\tau \to \infty$. It should be remembered that we are writing the arguments of the correlation functions as subscripts, e.g., $-\psi_\tau''$ is really

$$-\psi''(\tau) = 4\pi^2 \int_0^\infty f^2 w(f) \cos 2\pi f\tau \, df \quad (3.3\text{–}8)$$

As τ becomes larger and larger the behavior of I at τ is influenced less and less by the fact that it goes through zero with a positive slope at $\tau = 0$. Hence (3.4–1) should approach the probability that, for any interval of length $d\tau$ chosen at random, I will go through zero with a negative slope. Because of symmetry, this is half the probability that it will go through zero. Thus (3.4–1) should approach, from (3.3–10),

$$\frac{d\tau}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} \quad (3.4\text{–}3)$$

as $\tau \to \infty$. It actually does this since M approaches a diagonal matrix and both M_{23} and H approach zero with $M_{23}/H \to M_{22} \to -\psi_0^2\psi_0''$. For a low pass filter cutting off at f_b (3.4–3) is

$$d\tau f_b 3^{-1/2} \quad (3.4\text{–}4)$$

The behavior of (3.4–1) as $\tau \to 0$ is quite a bit more difficult to work out.

M_{22} and M_{23} go to zero as τ^4, $M_{22}^2 - M_{23}^2$ as τ^{10}, and consequently H goes to infinity as τ^{-1}. The final result is that (3.4–1) approaches

$$d\tau \frac{\tau}{8}\left[\frac{\psi_0\psi_0^{(4)} - \psi_0''^2}{-\psi_0\psi_0''}\right] \tag{3.4–5}$$

as $\tau \to 0$, assuming $\psi^{(4)}$ exists. Here the superscript (4) indicates the fourth derivative at $\tau = 0$,

$$\psi_0^{(4)} = 16\pi^4 \int_0^\infty f^4 w(f)\, df \tag{3.4–6}$$

For a low pass filter cutting off at f_b (3.4–5) is

$$d\tau \frac{\tau}{30}(2\pi f_b)^2 \tag{3.4–7}$$

When (3.4–1) is applied to a low pass filter, it turns out that instead of τ the variable

$$\varphi = 2\pi f_b \tau, \qquad d\varphi = 2\pi f_b\, d\tau \tag{3.4–8}$$

is more convenient to handle. Thus, if we write (3.4–1) as $p(\varphi)\, d\varphi$, it follows from (3.4–4) and (3.4–7) that

$$\begin{aligned} p(\varphi) &\to \frac{1}{2\pi\sqrt{3}} = .0919 \quad \text{as} \quad \varphi \to \infty \\ p(\varphi) &\to \frac{\varphi}{30} \quad \text{as} \quad \varphi \to \infty \end{aligned} \tag{3.4–9}$$

$p(\varphi)$ has been computed and plotted on Fig. 1 as a function of φ for the range 0 to 9. From the curve and the theory it is evident that beyond 9 $p(\varphi)$ oscillates about 0.0919 with ever decreasing amplitude.

We may take $p(\varphi)\, d\varphi$ to be the probability that I goes through zero in $\varphi, \varphi + d\varphi$, when it is known that I goes through zero at $\varphi = 0$ with a slope opposite to that at φ. $p(\varphi)\, d\varphi$ exceeds the probability that I goes through zero at $\varphi = 0$ and in $\varphi, \varphi + d\varphi$ with no zeros in between. This is because $p(\varphi)\, d\varphi$ includes all curves of the latter class and in addition those which may have an even number of zeros between 0 and φ. From this it follows that the curve giving the probability density of the intervals between zeros must be underneath the curve of $p(\varphi)$.

A partial check on the curve for $p(\varphi)$ may be obtained by comparing it with a probability density function obtained experimentally by M. E. Campbell for the intervals between 754 successive zeros. He passed thermal noise through a band pass filter, the lower cutoff being around 200 cps and the upper cutoff being around 3000 cps. The upper cutoff was rather grad-

ual and it is difficult to assign a representative value. The crosses on figure 1 are obtained from his data when we assume that his filter behaves like a low pass filter with a cutoff at $f_b = 2850$, this choice being made in order to make the maximum of his curve coincide with that of $p(\varphi)$.

It is seen that some of the crosses lie above $p(\varphi)$. This is probably due to the fact that the actual filter differs somewhat from the assumed low pass filter.

On Fig. 1 there is also plotted a function closely related to (3.4–1). It is the low pass filter form of the following: The probability of I passing

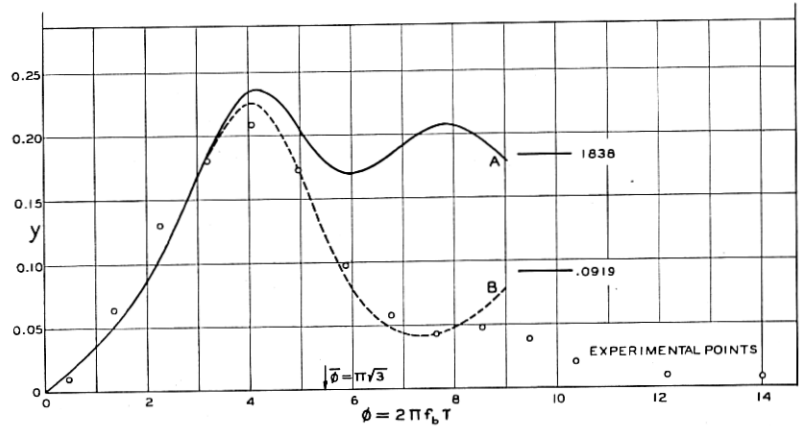

Fig. 1—Distribution of intervals between zeros—low-pass filter
$y_A \Delta\varphi$ is probability of a zero in $\Delta\varphi$ when a zero is at origin.
$y_B \Delta\varphi$ is probability of a zero in $\Delta\varphi$ when a zero is at origin and slopes at zeros are of opposite signs.
$y_B = p(\varphi), f_b =$ filter cutoff, $\tau =$ time between zeros.

through zero in τ, $\tau + d\tau$ when it is known that I passes through zero at $\tau = 0$ is

$$\frac{d\tau}{\pi}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2}\left[\frac{M_{23}}{H}\right](\psi_0^2 - \psi_\tau^2)^{-3/2}[1 + H \tan^{-1} H] \qquad (3.4\text{–}10)$$

where the notation is the same as in (3.4–1) and $-\frac{\pi}{2} \leq \tan^{-1} H \leq \frac{\pi}{2}$. This curve should always lie above $p(\varphi)$ and the small difference between the curves out to $\varphi = 4$ indicates that the true distribution of zeros is given closely by $p(\varphi)$ out to this point.

When (3.4–1) is applied to a relatively narrow band pass filter or some similar device we may make some approximations and obtain an expression somewhat simpler than (3.4–1). As a guide we consider our usual ideal

band pass filter whose range extends from f_a to f_b. The correlation function is given by (3.2–5).

$$\begin{aligned} \psi_\tau &= \frac{w_0}{\pi\tau} \sin \pi\tau(f_b - f_a) \cos \pi\tau(f_b + f_a) \\ \psi_0 &= w_0(f_b - f_a) \end{aligned} \tag{3.2–5}$$

From physical considerations we know that in a narrow filter most of the distances between zeros will be nearly equal to

$$\tau_1 = \frac{1}{f_b + f_a}$$

i.e., nearly equal to the distance between the zeros of a sine wave having the mid-band frequency. We therefore expect (3.4–1) to have a peak very close to τ_1. We also expect peaks at $3\tau_1$, $5\tau_1$ etc. but we shall not consider these. We wish to examine the behavior of (3.4–1) near τ_1.

It turns out that M_{23} is nearly equal to M_{22} so that H is large and (3.4–1) becomes approximately

$$\frac{d\tau}{2}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2} \frac{M_{23}}{[\psi_0^2 - \psi_\tau^2]^{3/2}}$$

where τ is near τ_1.

In order to see that M_{23} is nearly equal to M_{22} we use the expressions

$$\begin{aligned} M_{22} &= -\psi_0''(\psi_0^2 - \psi_\tau^2) - \psi_0\psi_\tau'^2 \\ M_{23} &= \psi_\tau''(\psi_0^2 - \psi_\tau^2) + \psi_\tau\psi_\tau'^2 \\ M_{22} + M_{23} &= (\psi_0 - \psi_\tau)[(\psi_0 + \psi_\tau)(\psi_\tau'' - \psi_0'') - \psi_\tau'^2] \\ &= (\psi_0 - \psi_\tau)[B + C] \\ M_{22} - M_{23} &= (\psi_0 + \psi_\tau)[(\psi_0 - \psi_\tau)(-\psi_\tau'' - \psi_0'') - \psi_\tau'^2] \\ &= (\psi_0 + \psi_\tau)[-B + C] \\ B &= \psi_0\psi_\tau'' - \psi_\tau\psi_0'' \\ C &= -\psi_0\psi_0'' + \psi_\tau\psi_\tau'' - \psi_\tau'^2 \end{aligned}$$

From (3.2–5) it is seen that ψ_τ may be written as

$$\psi_\tau = A \cos \beta\tau, \qquad \beta = \pi(f_b + f_a)$$

where $\beta\tau_1 = \pi$ and A is a function of τ which varies slowly in comparison with $\cos \beta\tau$. We see that near τ_1, ψ_τ is nearly equal to $-\psi_0$. Likewise

ψ_τ' hovers around zero and ψ_τ'' is nearly equal to $-\psi_0''$. Differentiating with respect to τ gives

$$\psi_\tau' = A' \cos \beta\tau - A\beta \sin \beta\tau$$
$$\psi_\tau'' = (A'' - A\beta^2) \cos \beta\tau - 2A'\beta \sin \beta\tau$$
$$\psi_0'' = A_0'' - A_0\beta^2, \qquad \psi_0 = A_0$$

where A_0 and A_0'' are the values of A and its second derivative at τ equal to zero. These lead to

$$B = (A_0A'' - AA_0'') \cos \beta\tau - 2A_0A'\beta \sin \beta\tau$$
$$C = (AA'' - A'^2) \cos^2 \beta\tau - A_0A_0'' + (A_0^2 - A)^2\beta^2$$

We wish to show that $C + B$ and $C - B$ are of the same order of magnitude. If we can do this, it follows that $M_{22} - M_{23}$ is much smaller than $M_{22} + M_{23}$ since $\psi_0 - \psi_{\tau_1}$ is approximately $2\psi_0$ while $\psi_0 + \psi_{\tau 1}$ is quite small. Consequently we will have shown that M_{23} is nearly equal to M_{22}.

So far we have made no approximations. We now express the slowly varying function A as a power series in τ. Since ψ_0' and ψ_0''' must be zero for the type of functions we consider, it follows that

$$A = A_0 + \frac{\tau^2}{2} A_0'' + \cdots$$
$$A' = \tau A_0'' + \cdots$$
$$A'' = A_0'' + \frac{\tau^2}{2} A_0^{(4)} + \cdots$$

where we neglect all powers higher than the second. Multiplication and squaring gives

$$A^2 - A_0^2 = \tau^2 A_0 A_0''$$
$$AA'' - A'^2 = A_0 A_0'' + \frac{\tau^2}{2} (A_0 A_0^{(4)} - A_0''^2)$$
$$= A_0 A_0'' + F$$
$$A_0 A'' - AA_0'' = \frac{\tau^2}{2} (A_0 A_0^{(4)} - A_0''^2) = F$$

Since, for small τ, A and A'' are nearly equal to A_0 and A_0'', respectively we see that the difference on the left is small relative to A_0A_0'', i.e.,

$$|F| << |A_0A_0''|$$

Our expression for B and C become approximately

$$B = F \cos \beta\tau - 2A_0 A_0'' \beta\tau \sin \beta\tau$$

$$C = F \cos^2 \beta\tau - A_0 A_0'' \sin^2 \beta\tau - A_0 A_0'' \beta^2 \tau^2$$

When τ is near τ_1, $\beta\tau$ is approximately π. Hence both $C + B$ and $C - B$ are approximately $-A_0 A_0'' \pi^2$ and are of the same order of magnitude. Consequently M_{22} and M_{23} are both nearly equal and

$$M_{23} = \psi_0[C + B]$$

$$= -A_0^2 A_0'' \pi^2$$

When this expression for M_{23} is used our approximation to (3.4–1) gives us the result: If the correlation function is of the form

$$\psi_\tau = A \cos \beta\tau$$

where A is a slowly varying function of τ, the probability that the distance between two successive zeros lies between τ and $\tau + d\tau$ is approximately

$$\frac{d\tau}{2} \frac{a}{[1 + a^2(\tau - \tau_1)^2]^{3/2}}$$

where a is positive and

$$a^2 = \frac{A_0 \beta^2}{-A_0'' \tau_1^2}, \qquad \tau_1 = \frac{\pi}{\beta}$$

For our ideal band pass filter with the pass band $f_b - f_a$,

$$a = \sqrt{3}\, \frac{(f_b + f_a)^2}{f_b - f_a}, \qquad \tau_1 = \frac{1}{f_b + f_a}$$

and the average value of $|\tau - \tau_1|$ is a^{-1}. Thus

$$\frac{\text{ave.}\,|\tau - \tau_1|}{\tau_1} = \frac{1}{a\tau_1} = \frac{f_b - f_a}{\sqrt{3}\,(f_b + f_a)} = \frac{1}{2\sqrt{3}} \frac{\text{band width}}{\text{mid-frequency}}$$

When the correlation function cannot be put in the form assumed above but still behaves like a sinusoidal wave with slowly varying amplitude we may use our first approximation to (3.4–1). Thus, the probability that the distance between two successive zeros lies between τ and $\tau + d\tau$ is approximately

$$\frac{b\, d\tau}{[\psi_0^2 - \psi_\tau^2]^{3/2}}$$

when τ lies near τ_1 where τ_1 is the smallest value of τ which makes ψ_τ approximately equal to $-\psi_0$. This probability is supposed to approach

zero rapidly as τ departs from τ_1, and b is chosen so that the integral over the effective region around τ_1 is unity.

It seems to be especially difficult to get an expression for the distribution of zeros for large spacing. One method, suggested by Prof. Goudsmit, is to amend the conditions leading to (3.4–1) by adding conditions that I be positive at equally spaced points along the time axis between 0 and τ. This leads to integrals which are hard to evaluate. For one point between 0 and τ the integral is of the form (3.5–7).

Another method of approach is to use the method of "in and exclusion" of zeros between 0 and τ. Consider the class of curves of I having a zero at $\tau = 0$. Then, in theory, our methods will allow us to compute the functions $p_0(\tau)$, $p_1(r, \tau)$, $p_2(r, s, \tau)$, associated with this class where

$p_0(\tau)\, d\tau$ is probability of curve having zero in $d\tau$

$p_1(r, \tau)\, d\tau\, dr$ is probability of curve having zeros in $d\tau$ and dr

$p_2(r, s, \tau)\, d\tau\, dr\, ds$ is probability of curve having zeros in $d\tau$, dr, and ds

In fact $p_0(\tau)\, d\tau$ is expression (3.4–10). The method of in and exclusion then leads to an expression for $P_0(\tau)\, d\tau$, the probability of having a zero at 0 and a zero in τ, $\tau + d\tau$ but none between 0 and τ. It is

$$P_0(\tau) = p_0(\tau) - \frac{1}{1!}\int_0^\tau p_1(r, \tau)\, dr + \frac{1}{2!}\int_0^\tau\int_0^\tau p_2(r, s, \tau)\, dr\, ds - \frac{1}{3!}\int_0^\tau\int_0^\tau\int_0^\tau p_3(r, s, t, \tau)\, dr\, ds\, dt + \cdots \tag{3.4–11}$$

Here again we run into difficult integrals. Incidentally, (3.4–11) may be checked for events occurring independently at random. Thus if $\nu\, d\tau$ is the probability of an event happening in $d\tau$, then, if ν is a constant and the events are independent, we have p_0, p_1, p_2, $\cdots$ given by ν, ν^2, ν^3, $\cdots$. From (3.4–11) we obtain the known result $P_0(\tau) = \nu e^{-\nu\tau}$.

We shall now derive (3.4–1). The work is based upon a generalization of (3.3–5): If y is a random curve described by (3.3–1), the probability that y will pass through zero in x_1, $x_1 + dx_1$ with a positive slope and through zero in x_2, $x_2 + dx_2$ with a negative slope is

$$-dx_1\, dx_2 \int_0^{+\infty} d\eta_1 \int_{-\infty}^{0} d\eta_2\, \eta_1\, \eta_2\, p(0, \eta_1, x_1; 0, \eta_2, x_2) \tag{3.4–12}$$

where $p(\xi_1, \eta_1, x_1; \xi_2, \eta_2, x_2)$ is the probability density function for the four random variables

$$\xi_i = F(a_1, a_2, \cdots, a_N; x_i)$$

$$\eta_i = \left[\frac{\partial F}{\partial x}\right]_{x=x_i}, \qquad i = 1, 2.$$

The x_1 and x_2 play the role of parameters in (3.4–12). This result may be established in much the same way as (3.3–5).

When we identify F with one of our representations, (2.8–1) or (2.8–6), of the noise current $I(t)$ it is seen that p is normal in four dimensions. We may obtain the second moments directly from this representation, as has been done in the equations just below (3.3–7). The same results may be obtained from the definition of $\psi(\tau)$, and for the sake of variety we choose this second method. We set $x_1 = t_1$, $x_2 = t_1 + \tau$. Then

$$\begin{aligned} \overline{\xi_1^2} &= \overline{\xi_2^2} = \overline{I^2(t)} = \psi_0 \\ \overline{\xi_1 \xi_2} &= \overline{I(t)I(t+\tau)} = \psi_\tau \\ \overline{\eta_1 \eta_2} &= \overline{\left(\frac{\partial I}{\partial t}\right)_t \left(\frac{\partial I}{\partial t}\right)_{t+\tau}} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I'(t+\tau)I'(t)\,dt \end{aligned} \tag{3.4–13}$$

where primes denote differentiation with respect to the arguments. Integrating by parts:

$$\int_0^T I'(t+\tau)\,dI(t) = [I'(t+\tau)I(t)]_0^T - \int_0^T I''(t+\tau)I(t)\,dt$$

We assume that I and its derivative remains finite so that the integrated portion vanishes, when divided by T, in the limit. Since

$$I''(t+\tau) = \frac{\partial^2}{\partial \tau^2} I(t+\tau)$$

we have

$$\overline{\eta_1 \eta_2} = -\frac{\partial^2}{\partial \tau^2}\psi(\tau) = -\psi_\tau''$$

Setting $\tau = 0$ gives

$$\overline{\eta_1^2} = \overline{\eta_2^2} = -\psi_0''$$

in agreement with the value of μ_{22} obtained from (3.3–7). In the same way

$$\begin{aligned} \overline{\xi_1 \eta_2} &= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I'(t+\tau)I(t)\,dt = \frac{\partial}{\partial \tau}\psi(\tau) \\ &= \psi_\tau' \\ \overline{\xi_2 \eta_1} &= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I'(t)I(t+\tau)\,dt \\ &= \text{“}\quad (-)\frac{1}{T}\int_0^T I'(t+\tau)I(t)\,dt \\ &= -\psi_\tau' \end{aligned}$$

where we have integrated by parts in getting $\overline{\xi_2\eta_1}$. Setting $\tau = 0$ and using $\psi_0' = 0$ gives

$$\overline{\xi_1\eta_1} = \overline{\xi_2\eta_2} = 0$$

In order to obtain the matrix M of the second moments μ_{rs} in a form fairly symmetrical about its center we choose the 1, 2, 3, 4 order of our variables to be ξ_1, η_1, η_2, ξ_2. From equations (3.4–13) etc. it is seen that this choice leads to the expression (3.4–2) for M.

When we put ξ_1 and ξ_2 equal to zero, we obtain for the probability density function in (3.4–12) the expression

$$\frac{|M|^{-1/2}}{4\pi^2} \exp\left[-\frac{1}{2|M|}(M_{22}\eta_1^2 + 2M_{23}\eta_1\eta_2 + M_{33}\eta_2^2)\right]$$

Because of the symmetry of M, M_{22} is equal to M_{33}. When, in the integral (3.4–12) we make the change of variable

$$x = \left[\frac{M_{22}}{2|M|}\right]^{1/2}\eta_1, \qquad y = -\left[\frac{M_{22}}{2|M|}\right]^{1/2}\eta_2$$

we obtain

$$\frac{dx_1\,dx_2}{\pi^2}\frac{|M|^{3/2}}{M_{22}^2}\int_0^\infty x\,dx\int_0^\infty dy\,ye^{-x^2-y^2+2(M_{23}/M_{22})xy}$$

The double integral may be evaluated by (3.5–4). Let

$$\varphi = \cos^{-1}\left(-\frac{M_{23}}{M_{22}}\right) = \cot^{-1}(-H), \qquad H = M_{23}[M_{22}^2 - M_{23}^2]^{-1/2}$$

where H is the same as that given in (3.4–2). Our expression now becomes

$$\frac{dx_1\,dx_2}{4\pi^2}\frac{|M|^{3/2}}{M_{22}^2 - M_{23}^2}[1 + H\cot^{-1}(-H)]$$

From a property of determinants

$$M_{22}M_{33} - M_{23}^2 = |M|(\psi_0^2 - \psi_\tau^2)$$

Using this to eliminate $|M|$ and dividing by

$$\frac{dx_1}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2}$$

which, from (3.3–10), is the probability of going through zero in x_1, $x_1 + dx_1$ with positive slope, gives the probability of going through zero in dx_2 with

negative slope when it is known that I goes through zero at x_1 with positive slope:

$$\frac{dx_2}{2\pi}\left[\frac{\psi_0}{-\psi_0''}\right]^{1/2}[M_{22}^2 - M_{23}^2]^{1/2}(\psi_0^2 - \psi_\tau^2)^{-3/2}[1 + H\cot^{-1}(-H)]$$

This is the same as (3.4–1).

The expression (3.4–10) is the same as the probability of I going through zero in $d\tau$ when it is known that I goes through zero at the origin with positive slope. This second probability may be obtained from (3.4–1) by adding the probability that I goes through $d\tau$ with positive slope when it is known to go through zero with positive slope. Thus we must add the expression containing the integral in which the integration in both η_1 and η_2 run from 0 to ∞. In terms of x and y this integral is

$$\int_0^\infty x\,dx \int_0^\infty dy\, ye^{-x^2-y^2-2(M_{23}/M_{22})xy}$$

This is equivalent to a change in the sign of M_{23} and hence of H. After this addition we must consider

$$\begin{aligned} &1 + H\cot^{-1}(-H) + 1 - H\cot^{-1} H \\ &= 2 + H[\cot^{-1}(-H) - \cot^{-1} H] \\ &= 2 + H[\pi - 2\cot^{-1} H] \\ &= 2[1 + H\tan^{-1} H] \end{aligned}$$

and this leads to (3.4–10).

3.5 Multiple Integrals

We wish to evaluate integrals of the form

$$J = \int_0^\infty dx_1 \int_0^\infty dx_2\, e^{-x_1^2 - 2ax_1x_2 - x_2^2} \qquad (3.5\text{–}1)$$

Our method of procedure is to first reduce the exponent to the sum of squares by a suitable linear change of variable and then change to polar coordinates. This method appears to work also for triple integrals of the same sort, but when it is applied to a four-fold integral, the last integration apparently cannot be put in closed form.

The reduction of the exponent to the sum of squares is based upon the transformation: If*

$$\begin{aligned} x_1 &= h_1 y_1 + h_2 D_{21} y_2 + h_3 D_{31} y_3 + \cdots + h_n D_{n,1} y_n \\ x_2 &= 0 \quad + h_2 D_{22} y_2 + \cdots \quad + h_n D_{n,2} y_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= 0 \quad + 0 \quad + \cdots + 0 \quad + h_n D_{n,n} y_n \end{aligned} \qquad (3.5\text{–}2)$$

* T. Fort, Am. Math. Monthly, 43 (1936), pp. 477–481. See also Scott and Mathews, Theory of Determinants, Cambridge (1904), Prob. 63, p. 276.

where $D_0 = 1$, $D_1 = a_{11}$, $D_{r,r} = D_{r-1}$, and D_{rs} is the cofactor of a_{sr} (or of a_{rs} because they are equal) in D_r :

$$D_r = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{12} & a_{22} & & \\ a_{1r} & \cdots & & a_{rr} \end{vmatrix}, \qquad h_r = [D_{r-1}D_r]^{-1/2},$$

then, if none of the D_r's is zero,

$$\sum_1^n a_{rs}\, x_r\, x_s = y_1^2 + y_2^2 + \cdots + y_n^2$$

From (3.5–2); the Jacobian $\partial(x_1, \cdots x_n)/\partial(y_1, \cdots y_n)$ is equal to $D_n^{-1/2}$.

Applying our transformation to the exponent:

$$x_1 = y_1 - aD_2^{-1/2}y_2$$

$$x_2 = 0 + D_2^{-1/2}y_2$$

$$D_2 = 1 - a^2$$

Since x_2 runs from 0 to ∞ so must y_2. The expression for x_1 shows that y_1 runs from $a\, D_2^{-1/2}y_2$ to ∞. The integral is therefore

$$J = D_2^{-1/2} \int_0^\infty dy_2 \int_{a D_2^{-1/2} y_2}^\infty e^{-y_1^2 - y_2^2}\, dy_1$$

We now change to polar coordinates:

$$y_1 = \rho \cos\theta$$
$$y_2 = \rho \sin\theta \qquad dy_1\, dy_2 = \rho\, d\rho\, d\theta$$

$$y_2 \geq 0 \text{ gives } 0 \leq \theta \leq \pi$$

$$y_1 \geq aD_2^{-1/2}y_2 \text{ gives } \cot\theta \geq aD_2^{-1/2}$$

and obtain

$$J = D_2^{-1/2} \int_0^{\cot^{-1} aD_2^{-1/2}} d\theta \int_0^\infty \rho e^{-\rho^2}\, d\rho$$

$$= \tfrac{1}{2} D_2^{-1/2} \cot^{-1}(aD_2^{-1/2})$$

where the arc-cotangent lies between 0 and π. This may be written in the simpler form

$$J = \tfrac{1}{2}(1 - a^2)^{-1/2} \cos^{-1} a = \tfrac{1}{2}\varphi \csc\varphi$$

where

$$a = \cos\varphi,$$

it being understood that $0 \leq \varphi \leq \pi$.

Other integrals may be obtained by differentiation. Thus from

$$\int_0^\infty dx \int_0^\infty dy\, e^{-x^2-y^2-2xy\cos\varphi} = \tfrac{1}{2}\varphi \csc\varphi \tag{3.5-3}$$

we obtain

$$\int_0^\infty dx \int_0^\infty dy\, xy\, e^{-x^2-y^2-2xy\cos\varphi} = \tfrac{1}{4}\csc^2\varphi(1 - \varphi\cot\varphi) \tag{3.5-4}$$

By using the same transformation we may obtain

$$\int_0^\infty dx \int_0^\infty dy\, ye^{-x^2-y^2-2axy} = \frac{\sqrt{\pi}}{4}\frac{1}{1+a} \tag{3.5-5}$$

Of course, we may expand part of the exponential in a power series and integrate termwise but this leads to a series which has to be summed in each particular case:

$$\int_0^\infty dx \int_0^\infty dy\, x^n y^m e^{-x^2-y^2-2axy}$$

$$= \frac{1}{4}\sum_{r=0}^{\infty}\frac{(-2a)^r}{r!}\Gamma\left(\frac{n+r+1}{2}\right)\Gamma\left(\frac{m+r+1}{2}\right)$$

If we take $-1 < R(m) < -\frac{1}{2}$, $-1 < R(m) < -\frac{1}{2}$, the series may be summed when $a = 1$. The result stated just below equation (3.8–9) is obtained by continuing m and n analytically.

The same methods will work when the limits are $\pm\infty$. We obtain, when m and n are integers,

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, x^n y^m e^{-x^2-y^2-2xy\cos\varphi}$$

$$= \begin{cases} 0, & n+m \text{ odd} \\ (-)^n\sqrt{\pi}\,\dfrac{\Gamma\left(\dfrac{m+n+1}{2}\right)}{(\sin\varphi)^{n+m+1}} & \\ \quad F\left(-n, -m; \dfrac{1-n-m}{2}; \dfrac{1-\cos\varphi}{2}\right), & n+m \text{ even} \end{cases} \tag{3.5-6}$$

The hypergeometric function may also be written as

$$F\left(-\frac{n}{2}, -\frac{m}{2}; \frac{1-n-m}{2}; \sin^2\varphi\right)$$

By transformations of this we are led to the following expression for the integral

$$0, \; n+m \text{ odd},$$

$$\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{(\sin\varphi)^{n+m+1}} F\left(-\frac{n}{2}, -\frac{m}{2}, \frac{1}{2}; \cos^2\varphi\right), \quad m, n \text{ both even},$$

$$-2\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(1+\frac{m}{2}\right)}{(\sin\varphi)^{n+m+1}} \cos\varphi F\left(\frac{1-m}{2}, \frac{1-n}{2}; \frac{3}{2}; \cos^2\varphi\right),$$

$$m, n \text{ odd}$$

As was mentioned earlier, the method used to evaluate the double integrals may also be applied to similar triple integrals. Here we state two results obtained in this way.

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \exp\left[-x^2 - y^2 - z^2 - 2cxy - 2bzx - 2ayz\right]$$
$$= \frac{1}{4}\left[\frac{\pi}{D_3}\right]^{1/2}[\alpha + \beta + \gamma - \pi]$$
$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \, yz \exp\left[-x^2 - y^2 - z^2 - 2cxy - 2bzx - 2ayz\right]$$
$$= \frac{\sqrt{\pi}}{8D_3}\left[\frac{1+a-b-c}{1+a} - \frac{a-bc}{D_3^{1/2}}(\alpha+\beta+\gamma-\pi)\right] \qquad (3.5\text{–}7)$$

where β and γ are obtained by cyclic permutation of a, b, c from

$$\alpha = \cos^{-1}\frac{a-cb}{(1-c^2)^{1/2}(1-b^2)^{1/2}} = \sin^{-1}\left[\frac{D_3}{(1-c^2)(1-b^2)}\right]^{1/2}$$
$$= \cot^{-1}\frac{a-bc}{D_3^{1/2}}$$

where α, β, γ all lie in the range 0, π and where

$$D_3 = \begin{vmatrix} 1 & c & b \\ c & 1 & a \\ b & a & 1 \end{vmatrix} = 1 + 2abc - a^2 - b^2 - c^2$$

For reference we state the integrals which arise from the definition of the normal distribution given in section (2.9)

$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n \exp\left[-\sum_1^n a_{rs}x_r x_s\right] = \left[\frac{\pi^n}{|a|}\right]^{1/2}$$
$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n \, x_t x_u \exp\left[-\sum^n a_{rs}x_r x_s\right] = \left[\frac{\pi^n}{|a|^3}\right]^{1/2}\frac{A_{tu}}{2} \qquad (3.5\text{–}8)$$

where the quadratic form is positive definite and $|a|$ is its determinant. A_{tu} is the cofactor of a_{tu}. Incidentally, these may be regarded as special cases of

$$\int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_n f\left(\sum_1^n a_{rs} x_r x_s\right) F\left(\sum_1^n b_r x_r\right)$$

$$= \frac{2}{\Gamma\left(\frac{n-1}{2}\right)} \left[\frac{\pi^{n-1}}{|a|}\right]^{1/2} \int_{-\infty}^{+\infty} dx \int_0^{\infty} dy\, y^{n-2} f(x^2 + y^2) \tag{3.5-9}$$

$$F\left\{x\left[\frac{\sum_1^n A_{rs} b_r b_s}{|a|}\right]^{1/2}\right\},$$

which is a generalization of a result given by Schlömilch.*

3.6 Distribution of Maxima of Noise Current

Here we shall use a result similar to those used in sections 3.3 and 3.4. Let y be a random curve given by (3.3–1),

$$y = F(a_1 \cdots a_N ; x). \tag{3.3-1}$$

If suitable conditions are satisfied, the probability that y has a maximum in the rectangle $(x_1, x_1 + dx_1, y_1, y_1 + dy_1)$, dx_1 and dy_1 being of the same order of magnitude, is[32]

$$-dx_1\, dy_1 \int_{-\infty}^{0} p(y_1, 0, \zeta)\zeta\, d\zeta \tag{3.6-1}$$

and the expected number of maxima of y in $a \leq x \leq b$ is obtained by integrating this expression over the range $-\infty \leq y_1 \leq \infty$, $a \leq x_1 \leq b$. $p(\xi, \eta, \zeta)$ is the probability density function for the random variables

$$\begin{aligned} \xi &= F(a_1, \cdots, a_N ; x_1) \\ \eta &= \left(\frac{\partial F}{\partial x}\right)_{x=x_1} \\ \zeta &= \left(\frac{\partial^2 F}{\partial x^2}\right)_{x=x_1} \end{aligned} \tag{3.6-2}$$

* Höheren Analysis, Braunschweig (1879), Vol. 2, p. 494, equ. (29).

[32] *Am. Jour. Math.*, Vol. 61 (1939) 409–416. A similar problem has been studied by E. L. Dodd, The Length of the Cycles Which Result From the Graduation of Chance Elements, Ann. Math. Stat., Vol. 10 (1939) 254–264. He gives a number of references to the literature dealing with the fluctuations of time series.

In our application of this result we replace x and y by t and I as before. Then

$$\xi = I = \sum_1^N c_n \cos(\omega_n t - \varphi_n)$$

$$\eta = I'$$

$$\zeta = I''$$

where the primes denote differentiation with respect to t. According to the central limit theorem the distribution of ξ, η, ζ approaches a normal law. The second moments defining this law may be obtained either from the above definitions of ξ, η, ζ, or may be obtained from the correlation function as was done in the work following equation (3.4–13).

$$\overline{\xi^2} = \psi_0, \qquad \overline{\eta^2} = -\psi_0'', \qquad \overline{\xi\eta} = 0$$

$$\overline{\eta\zeta} = \overline{I'(t)I''(t)} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I'(t)I''(t)\,dt$$

$$= \operatorname*{Limit}_{T\to\infty} \frac{1}{2T}[I'^2(T) - I'^2(0)] = 0$$

$$\overline{\xi\zeta} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I(t)I''(t)\,dt$$

$$= \operatorname*{Limit}_{\tau\to 0} \frac{\partial^2\psi(\tau)}{\partial\tau^2} = \psi_0''$$

$$\overline{\zeta^2} = \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I''(t)I''(t)\,dt$$

$$= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I^{(4)}(t)I(t)\,dt$$

$$= \psi_0^{(4)}$$

where the superscript (4) represents the fourth derivative. The matrix M of the moments is thus

$$M = \begin{bmatrix} \psi_0 & 0 & \psi_0'' \\ 0 & -\psi_0'' & 0 \\ \psi_0'' & 0 & \psi_0^{(4)} \end{bmatrix}$$

The determinant $|M|$ and the cofactors of interest are

$$|M| = -\psi_0''(\psi_0\psi_0^{(4)} - \psi_0''^2) \tag{3.6–3}$$

$$M_{11} = -\psi_0''\psi_0^{(4)}, \qquad M_{13} = \psi_0''^2, \qquad M_{33} = -\psi_0''\psi_0$$

The probability density function in (3.6–1) is

$$p(I, 0, \zeta) = (2\pi)^{-3/2} \mid M \mid^{-1/2} \exp \left[-\frac{1}{2 \mid M \mid} (M_{11} I^2 + M_{33} \zeta^2 + 2M_{13} I\zeta) \right] \tag{3.6-4}$$

and when this is put in (3.6–1) and the integration with respect to ζ performed we get

$$dI\, dt \frac{(2\pi)^{-3/2}}{M_{33}} \left[\mid M \mid^{1/2} e^{-M_{11} I^2/2\mid M \mid} + M_{13} I \left(\frac{\pi}{2M_{33}} \right)^{1/2} e^{-I^2/2\psi_0} \left(1 + \operatorname{erf} \frac{M_{13} I}{[2 \mid M \mid M_{33}]^{1/2}} \right) \right] \tag{3.6-5}$$

for the probability of a maximum occurring in the rectangle $dI\, dt$. As is mentioned just below expression (3.6–1), the expected number of maxima in the interval t_1, t_2 may be obtained by integrating (3.6–1) from t_1 to t_2 after replacing x by t, and I from $-\infty$ to $+\infty$ after replacing y by I. When we use (3.6–4) it is easier to integrate with respect to I first. The expected number is then

$$-\int_{t_1}^{t_2} dt \frac{M_{11}^{-1/2}}{2\pi} \int_{-\infty}^{0} \zeta \exp \left[-\frac{\zeta^2}{2 \mid M \mid} \left(M_{33} - \frac{M_{13}^2}{M_{11}} \right) \right] d\zeta = (t_2 - t_1) \frac{\psi_0^{(4)}}{2\pi} M_{11}^{-1/2} = \frac{t_2 - t_1}{2\pi} \left[\frac{\psi_0^{(4)}}{-\psi_0''} \right]^{1/2}$$

Hence the expected number of maxima per second is

$$\frac{1}{2\pi} \left[\frac{\psi_0^{(4)}}{-\psi_0''} \right]^{1/2} = \left[\frac{\int_0^\infty f^4 w(f)\, df}{\int_0^\infty f^2 w(f)\, df} \right]^{1/2} \tag{3.6-6}$$

For a band pass filter, the expected number of maxima per second is

$$\left[\frac{3}{5} \frac{f_b^5 - f_a^5}{f_b^3 - f_a^3} \right]^{1/2} \tag{3.6-7}$$

where f_b and f_a are the cut-off frequencies. Putting $f_a = 0$ so as to get a low pass filter,

$$f_b \left[\frac{3}{5} \right]^{1/2} = .775 f_b \tag{3.6-8}$$

From (3.6–8) and (3.6–5) we may obtain the probability density function for the maxima in the case of a low pass filter. Thus the probability that a maximum selected at random from the universe of maxima will lie in I, $I + dI$ is

$$\frac{dI}{3\sqrt{2\pi\psi_0}}\left[2e^{-9y^2/8} + \left(\frac{5\pi}{2}\right)^{1/2} ye^{-y^2/2}\left(1 + \operatorname{erf} y\left(\frac{5}{8}\right)^{1/2}\right)\right] \qquad (3.6\text{–}9)$$

where

$$y = \frac{I}{\sqrt{\psi_0}}$$

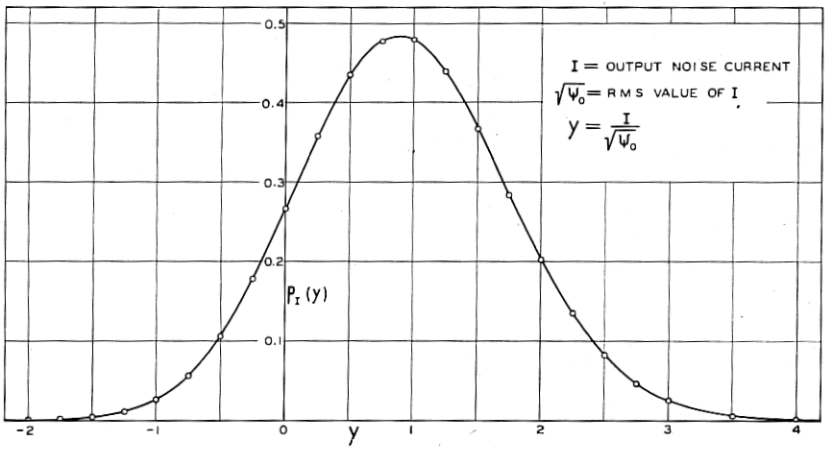

Fig. 2—Distribution of maxima of noise current. Noise through ideal low-pass filter. $\frac{p_I(y)}{\sqrt{\psi_0}}\,dI$ = probability that a maximum of I selected at random lies between I and $I + dI$.

When y is large and positive (3.6–9) is given asymptotically by

$$\frac{dI}{\sqrt{\psi_0}}\,\frac{\sqrt{5}}{3}\,ye^{-y^2/2}$$

If we write (3.6–9) as $p_I(y)\,dy$, the probability density $p_I(y)$ of y may be plotted as a function of y. This plot is shown in Fig. 2. The distribution function $P(I_{\max} < y\sqrt{\psi_0})$ defined by

$$P(I_{\max} < y\sqrt{\psi_0}) = \int_{-\infty}^{y} p_I(y)\cdot dy$$

and which gives the probability that a maximum selected at random is less than a specified $y\sqrt{\psi_0} = I$, is one of the four curves plotted in Fig. 4.

If I is large and positive we may obtain an approximation from (3.6–5). We observe that

$$\frac{M_{11}}{|M|} = \frac{\psi_0^{(4)}}{\psi_0\psi_0^{(4)} - \psi_0''^2} > \frac{1}{\psi_0}$$

so that when I is large and positive

$$e^{-M_{11}I^2/2|M|} \ll e^{-I^2/2\psi_0}$$

Also, in these circumstances the 1 + erf is nearly equal to two. Thus retaining only the important terms and using the definitions of the M's gives the approximation to (3.6–5):

$$\frac{dI\ dt}{2\pi\psi_0}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} Ie^{-I^2/2\psi_0} \tag{3.6–10}$$

From this it follows that the expected number of maxima per second lying above the line $I = I_1$ is approximately[33] when I_1 is large,

$$\frac{1}{2\pi}\left[\frac{-\psi_0''}{\psi_0}\right]^{1/2} e^{-I_1^2/2\psi_0} \tag{3.6–11}$$

$$= e^{-I_1^2/2\psi_0} \times \tfrac{1}{2}[\text{the expected number of zeros of } I \text{ per second}]$$

It is interesting to note that the approximation (3.6–11) for the expected number of maxima above I_1 is the same as the exact expression (3.3–14) for the expected number of times I will pass through I_1 with positive slope.

3.7 Results on the Envelope of the Noise Current

The noise current flowing in the output of a relatively narrow band pass filter has the character of a sine wave of, roughly, the midband frequency whose amplitude fluctuates irregularly, the rapidity of fluctuation being of the order of the band width. Here we study the fluctuations of the envelope of such a wave.

First we define the envelope. Let f_m be a representative midband frequency. Then if

$$\omega_m = 2\pi f_m \tag{3.7–1}$$

the noise current may be represented, see (2.8–6), by

$$\begin{aligned} I &= \sum_{n=1}^{N} c_n \cos(\omega_n t - \omega_m t - \varphi_n + \omega_m t) \\ &= I_c \cos \omega_m t - I_s \sin \omega_m t \end{aligned} \tag{3.7–2}$$

where the components I_c and I_s are

$$\begin{aligned} I_c &= \sum_{n=1}^{N} c_n \cos(\omega_n t - \omega_m t - \varphi_n) \\ I_s &= \sum_{n=1}^{N} c_n \sin(\omega_n t - \omega_m t - \varphi_n) \end{aligned} \tag{3.7–3}$$

[33] This expression agrees with an estimate made by V. D. Landon, *Proc. I. R. E.*, 29 (1941), 50–55. He discusses the number of crests exceeding four times the r.m.s. value of I. This corresponds to $I_1^2 = 16\psi_0$.

The envelope, R, is a function of t defined by

$$R = [I_c^2 + I_s^2]^{1/2} \tag{3.7-4}$$

It follows from the central limit theorem and the definitions (3.7–3) of I_c and I_s that these are two normally distributed random variables. They are independent since $\overline{I_c I_s} = 0$. They both have the same standard deviation, namely the square root of

$$\overline{I_c^2} = \overline{I_s^2} = \overline{I^2} = \int_0^\infty w(f)\, df = \psi_0 \tag{3.7-5}$$

Consequently, the probability that the point (I_c, I_s) lies within the elementary rectangle $dI_c dI_s$ is

$$\frac{dI_c\, dI_s}{2\pi\psi_0} \exp\left[-\frac{I_c^2 + I_s^2}{2\psi_0}\right] \tag{3.7-6}$$

In much of the following work it is convenient to introduce another random variable θ where

$$\begin{aligned} I_c &= R\cos\theta \\ I_s &= R\sin\theta \end{aligned} \tag{3.7-7}$$

Since I_c and I_s are random variables so are R and θ. The differentials are related by

$$dI_c dI_s = R d\theta dR \tag{3.7-8}$$

and the distribution function for R and θ is obtainable from (3.7–6) when the change of variables is made:

$$\frac{d\theta}{2\pi}\,\frac{R\,dR}{\psi_0}\, e^{-R^2/2\psi_0} \tag{3.7-9}$$

Since this may be expressed as a product of terms involving R only and θ only, R and θ are independent random variables, θ being uniformly distributed over the range 0 to 2π and R having the probability density[34]

$$\frac{R}{\psi_0} e^{-R^2/2\psi_0} \tag{3.7-10}$$

Expression (3.7–10) gives the probability density for the value of the envelope. Like the normal law for the instantaneous value of I, it depends only upon the average total power

$$\psi_0 = \int_0^\infty w(f)\, df$$

[34] See V. D. Landon and K. A. Norton, *I.R.E. Proc.*, 30 (1942), 425–429.

We now study the correlation between R at time t and its value at some later time $t + \tau$. Let the subscripts 1 and 2 refer to the times t and $t + \tau$, respectively. Then from (3.7–3) and the central limit theorem it follows that the four random variables I_{c1}, I_{s1}, I_{c2}, I_{s2} have a four dimensional normal distribution. This distribution is determined by the second moments

$$\overline{I_{c1}^2} = \overline{I_{s1}^2} = \overline{I_{c2}^2} = \overline{I_{s2}^2} = \psi_0 = \mu_{11}$$

$$\overline{I_{c1} I_{s1}} = \overline{I_{c2} I_{s2}} = 0$$

$$\overline{I_{c1} I_{c2}} = \overline{I_{s1} I_{s2}} = \frac{1}{2} \sum_{n=1}^{N} c_n^2 \cos (\omega_n \tau - \omega_m \tau)$$

$$\rightarrow \int_0^\infty w(f) \cos 2\pi(f - f_m)\tau \, df = \mu_{13} \tag{3.7–11}$$

$$\overline{I_{c1} I_{s2}} = -\overline{I_{c2} I_{s1}} = \frac{1}{2} \sum_{n=1}^{N} c_n^2 \sin (\omega_n \tau - \omega_m \tau)$$

$$\rightarrow \int_0^\infty w(f) \sin 2\pi(f - f_m)\tau \, df = \mu_{14}$$

The moment matrix for the variables in the order I_{c1}, I_{s1}, I_{c2}, I_{s2} is

$$M = \begin{bmatrix} \psi_0 & 0 & \mu_{13} & \mu_{14} \\ 0 & \psi_0 & -\mu_{14} & \mu_{13} \\ \mu_{13} & -\mu_{14} & \psi_0 & 0 \\ \mu_{14} & \mu_{13} & 0 & \psi_0 \end{bmatrix}$$

and from this it follows that the cofactors of the determinant $| M |$ are

$$\begin{aligned} M_{11} = M_{22} = M_{33} = M_{44} &= \psi_0(\psi_0^2 - \mu_{13}^2 - \mu_{14}^2) \\ &= \psi_0 A, \qquad A = \psi_0^2 - \mu_{13}^2 - \mu_{14}^2 \\ M_{12} = \quad M_{34} &= 0 \\ M_{13} = \quad M_{24} &= -\mu_{13} A \\ M_{14} = -M_{23} &= -\mu_{14} A \\ | M | &= A^2 \end{aligned} \tag{3.7–12}$$

The probability density of the four random variables is therefore

$$\frac{1}{4\pi^2 A} \exp - \frac{1}{2A} [\psi_0(I_1^2 + I_2^2 + I_3^2 + I_4^2)$$

$$- 2\mu_{13}(I_1 I_3 + I_2 I_4) - 2\mu_{14}(I_1 I_4 - I_2 I_3)]$$

where we have written I_1, I_2, I_3, I_4 for I_{c1}, I_{s1}, I_{c2}, I_{s2}. We now make the transformation

$$I_1 = R_1 \cos \theta_1 \qquad I_3 = R_2 \cos \theta_2$$

$$I_2 = R_1 \sin \theta_1 \qquad I_4 = R_2 \sin \theta_2$$

and average the resulting probability density over θ_1 and θ_2 in order to get the probability that R_1 and R_2 lie in dR_1 and dR_2. It is

$$\frac{R_1 R_2 \, dR_1 \, dR_2}{4\pi^2 A} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \exp$$

$$- \frac{1}{2A} [\psi_0 R_1^2 + \psi_0 R_2^2 - 2\mu_{13} R_1 R_2 \cos (\theta_2 - \theta_1) - 2\mu_{14} R_1 R_2 \sin (\theta_2 - \theta_1)]$$

Since the integrand is a periodic function of θ_2 we may integrate from $\theta_2 = \theta_1$ to $\theta_2 = \theta_1 + 2\pi$ instead of from 0 to 2π. This integration gives the Bessel function, I_0, of the first kind with imaginary argument. The resulting probability density for R_1 and R_2 is

$$\frac{R_1 R_2}{A} I_0 \left(\frac{R_1 R_2}{A} [\mu_{13}^2 + \mu_{14}^2]^{1/2} \right) \exp - \frac{\psi_0}{2A} (R_1^2 + R_2^2) \qquad (3.7\text{–}13)$$

where, from (3.7–12),

$$A = \psi_0^2 - \mu_{13}^2 - \mu_{14}^2$$

μ_{13} and μ_{14} are given by (3.7–11). Of course, R_1 and R_2 are always positive.

For an ideal band pass filter with cut-offs at f_a and f_b we set

$$f_m = \frac{f_b + f_a}{2}, \qquad w(f) = w_0 \quad \text{for} \quad f_a < f < f_b \quad ,$$

and obtain

$$\psi_0 = w_0(f_b - f_a)$$

$$\mu_{13} = \int_{f_a}^{f_b} w_0 \cos 2\pi(f - f_m)\tau \, df = \frac{w_0 \sin \pi(f_b - f_a)\tau}{\pi\tau}$$

$$\mu_{14} = \int_{f_a}^{f_b} w_0 \sin 2\pi(f - f_m)\tau \, df = 0$$

The I_0 term in (3.7–13), which furnishes the correlation between R_1 and R_2, becomes

$$I_0 \left(\frac{R_1 R_2}{\psi_0} \frac{\frac{\sin x}{x}}{1 - \frac{\sin^2 x}{x^2}} \right)$$

where x is $\pi(f_b - f_a)\tau$. When x is a multiple of π, R_1 and R_2 are independent random variables. When x is zero R_1 and R_2 are equal. Hence we may say, roughly, that the period of fluctuation of R is the time it takes x to increase from 0 to π or $(f_b - f_a)^{-1}$. This is related to the result given in the next section, namely that the expected number of maxima of the envelope is .641 $(f_b - f_a)$ per second.

3.8 Maxima of R

Here we wish to study the distribution of the maxima of R.* Our work is based upon the expression, cf. (3.6–1),

$$-dR\,dt \int_{-\infty}^{0} p(R, 0, R'')R''\,dR'' \tag{3.8–1}$$

for the probability that a maximum of R falls within the elementary rectangle $dR\,dt$. $p(R, R', R'')$ is the probability density for the three dimensional distribution of R, R', R'' where the primes denote differentiation with respect to t.

We shall determine $p(R, R', R'')$ from the probability density of I_c, I_s', I_c'', I_s, I_c', I_s'', which we shall denote by $x_1, x_2, \cdots x_6$. The interchange of I_s' and I_c' is suggested by the later work. It is convenient to introduce the notation

$$\begin{aligned} b_n &= (2\pi)^n \int_0^\infty w(f)(f - f_m)^n\,df \\ b_0 &= \psi_0 \end{aligned} \tag{3.8–2}$$

where f_m is the mid-band frequency, i.e., the frequency chosen in the definition of the envelope R. b_n is seen to be analogous to the derivatives of $\psi(\tau)$ at $\tau = 0$.

From the definitions (3.7–3) of I_c and I_s we obtain the second moments

$$\begin{aligned} \overline{x_1^2} &= \overline{I_c^2} = \psi_0 = b_0 \\ \overline{x_4^2} &= \overline{I_s^2} = b_0 \\ \overline{x_2^2} &= \overline{I_s'^2} = \sum_1^N w(f_n)\Delta f 4\pi^2(f_n - f_m)^2 = b_2 \\ \overline{x_5^2} &= \overline{I_c'^2} = b_2 \\ \overline{x_3^2} &= \overline{I_c''^2} = b_4 \\ \overline{x_6^2} &= \overline{I_s''^2} = b_4 \end{aligned}$$

* Incidentally, most of the analysis of this section was originally developed in a study of the stability of repeaters in a loaded telephone transmission line. The envelope, R, was associated with the "returned current" produced by reflections from line irregularities. However, the study fell short of its object and the only results which seemed worth salvaging at the time were given in reference[25] cited in Section 3.3.

$$\overline{x_1 x_2} = \overline{I_c I_s'} = \sum_1^N w(f_n)\Delta f 2\pi(f_n - f_m) = b_1$$

$$\overline{x_4 x_5} = \overline{I_s I_c'} = -b_1$$

$$\overline{x_1 x_3} = \overline{I_c I_c''} = -\sum_1^N w(f)\Delta f 4\pi^2(f_n - f_m)^2 = -b_2$$

$$\overline{x_4 x_6} = \overline{I_s I_s''} = -b_2$$

$$\overline{x_2 x_3} = \overline{I_s' I_c''} = -b_3$$

$$\overline{x_5 x_6} = \overline{I_c' I_s''} = b_3$$

All of the other second moments are zero. The moment matrix M is thus

$$M = \begin{bmatrix} b_0 & b_1 & -b_2 & 0 & 0 & 0 \\ b_1 & b_2 & -b_3 & 0 & 0 & 0 \\ -b_2 & -b_3 & b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_0 & -b_1 & -b_2 \\ 0 & 0 & 0 & -b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -b_2 & b_3 & b_4 \end{bmatrix}$$

The adjoint matrix is

$$\begin{bmatrix} B_0 & B_1 & -B_2 & 0 & 0 & 0 \\ B_1 & B_{22} & -B_3 & 0 & 0 & 0 \\ -B_2 & -B_3 & B_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_0 & -B_1 & -B_2 \\ 0 & 0 & 0 & -B_1 & B_{22} & B_3 \\ 0 & 0 & 0 & -B_2 & B_3 & B_4 \end{bmatrix}$$

$$B_0 = (b_2b_4 - b_3^2)B \qquad B_{22} = (b_0b_4 - b_2^2)B$$

$$B_1 = -(b_1b_4 - b_2b_3)B \qquad B_3 = -(b_0b_3 - b_1b_2)B$$

$$B_2 = (b_1b_3 - b_2^2)B \qquad B_4 = (b_0b_2 - b_1^2)B \qquad (3.8\text{–}3)$$

$$B = b_0b_2b_4 + 2\,b_1b_2b_3 - b_2^3 - b_0b_3^2 - b_4b_1^2$$

$$|M| = B^2$$

where B is the determinant of the third order matrices in the upper left and lower right corners of M.

As in the earlier work, the distribution of $x_1, \cdots, x_6$ is normal in six dimensions. The exponent is $-[2\,|\,M\,|\,]^{-1}$ times

$$B_0(x_1^2 + x_4^2) + 2B_1(x_1x_2 - x_4x_5) - 2B_2(x_1x_3 + x_4x_6) + B_{22}(x_2^2 + x_5^2) - 2B_3(x_2x_3 - x_5x_6) + B_4(x_3^2 + x_6^2) \qquad (3.8\text{–}4)$$

In line with the earlier work we set

$$x_1 = I_c = R\cos\theta \qquad x_4 = I_s = R\sin\theta$$

$$x_2 = I_s' = R'\sin\theta + R\cos\theta\theta'$$

$$x_5 = I_c' = R'\cos\theta - R\sin\theta\theta'$$

$$x_3 = I_c'' = R''\cos\theta - 2R'\sin\theta\theta' - R\cos\theta\theta'^2 - R\sin\theta\theta''$$

$$x_6 = I_s'' = R''\sin\theta + 2R'\cos\theta\theta' - R\sin\theta\theta'^2 + R\cos\theta\theta''$$

The angle θ varies from 0 to 2π and θ' and θ'' vary from $-\infty$ to $+\infty$. By forming the Jacobian it may be shown that

$$dx_1\,dx_2\cdots dx_6 = R^3\,dR\,dR'\,dR''\,d\theta\,d\theta'\,d\theta''$$

Also, the quantities in (3.8–4) are

$$x_1^2 + x_4^2 = R^2 \qquad x_1x_3 + x_4x_6 = RR'' - R^2\theta'^2$$

$$x_1x_2 - x_4x_5 = R^2\theta' \qquad x_2^2 + x_5^2 = R'^2 + R^2\theta'^2$$

$$x_2x_3 - x_5x_6 = RR''\theta' - 2R'^2\theta' - R'R\theta'' - R^2\theta'^3$$

$$x_3^2 + x_6^2 = R''^2 - 2RR''\theta'^2 + 4R'^2\theta'^2 + 4RR'\theta'\theta'' + R^2\theta'^4 + R^2\theta''^2$$

The expression for $p(R, 0, R'')$ is obtained when we set these values of the x's in (3.8–4) and integrate the resulting probability density over the ranges of θ, θ', θ'':

$$p(R, 0, R'') = \frac{R^3}{8\pi^3 B}\int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} d\theta' \int_{-\infty}^{+\infty} d\theta'' \qquad (3.8\text{–}5)$$

$$\exp -\frac{1}{2B^2}\,[B_0R^2 + 2B_1R^2\theta' - 2B_2(RR'' - R^2\theta'^2) + B_{22}R^2\theta'^2 - 2B_3R\theta'(R'' - R\theta'^2) + B_4(R''^2 - 2RR''\theta'^2 + R^2\theta'^4 + R^2\theta''^2)]$$

The integrations with respect to θ and θ'' may be performed at once leaving $p(R, 0, R'')$ expressed as a single integral which, unfortunately, appears to be difficult to handle. For this reason we assume that $w(f)$ is symmetrical about the mid-band frequency f_m. From (3.8–2), b_1 and b_3 are zero and from (3.8–3), B_1 and B_3 are zero.

With this assumption (3.8–5) yields

$$p(R, 0, R'') = R^2(2\pi)^{-3/2} D_4^{-1/2} \int_{-\infty}^{+\infty} d\theta' \tag{3.8-6}$$

$$\exp -\frac{1}{2B^2}[B_0R^2 + R([B_{22} + 2B_2]R\theta'^2 - 2B_2R'') + B_4(R'' - R\theta'^2)^2]$$

The probability that a maximum occurs in the elementary rectangle $dR\ dt$ is, from (3.8–1), $p(t, R)\ dR\ dt$ where

$$p(t, R) = -\int_{-\infty}^{0} p(R, 0, R'')R''\ dR'' \tag{3.8-7}$$

We put (3.8–6) in this expression and make the following change of variables.

$$\begin{aligned} x &= \frac{B_4^{1/2}}{\sqrt{2}\,B} R\theta'^2, \qquad y = -\frac{B_4^{1/2}}{\sqrt{2}\,B} R'' \\ z &= -\frac{B_2}{\sqrt{2B_4}\,B} R = \frac{b_2^2}{\sqrt{2B_4}} R \\ b &= -\frac{(B_{22} + 2B_2)}{2B\,b_2^2} = \left[\frac{3}{2} - \frac{b_0b_4}{2b_2^2}\right] = \tfrac{1}{2}(3 - a^2) \\ a^2 &= \frac{B_0}{2B^2}\frac{2B_4}{b_2^4} = \frac{b_0b_4}{b_2^2} \end{aligned} \tag{3.8-8}$$

where we have used the expressions for the B's obtained by setting b_1 and b_3 to zero in (3.8–3). Thus

$$p(t, R) = \frac{4}{b_0b_2^4}\left(\frac{Bz}{2\pi}\right)^{3/2} \int_0^\infty y\,dy \int_0^\infty x^{-1/2}\,dx \tag{3.8-9}$$

$$\exp\,[-a^2z^2 + 2bzx + 2zy - (x + y)^2]$$

As was to be expected, this expression shows that $p(t, R)$ is independent of t.

A series for $p(t, R)$ may be obtained by expanding $\exp 2z(y + bx)$ and then integrating termwise. We use

$$\int_0^\infty dy \int_0^\infty dx\, x^\mu y^\gamma e^{-(x+y)^2} = \frac{\sqrt{\pi}}{2^{\mu+\gamma+2}} \frac{\Gamma(\gamma + 1)\Gamma(\mu + 1)}{\Gamma\left(\frac{\mu + \gamma + 3}{2}\right)}$$

which may be evaluated by setting

$$x = \rho^2 \cos^2 \varphi, \qquad y = \rho^2 \sin^2 \varphi$$

The double integral in (3.8–9) becomes

$$e^{-a^2z^2}\sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(2z)^n}{n!}\sum_{m=0}^{n}\frac{n!\,b^m}{m!\,(n-m)!}\frac{\Gamma(m+\frac{1}{2})\Gamma(n-m+2)}{2^{n+2}\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}$$

$$=\pi 2^{-5/2}\sum_{n=0}^{\infty}\frac{z^n e^{-a^2z^2}}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}A_n$$

where $A_0 = 1$ and

$$A_n = \sum_{m=0}^{n}\frac{(\frac{1}{2})(\frac{3}{2})\cdots(m-\frac{1}{2})}{m!}(n-m+1)b^m, \qquad 0 < n \tag{3.8–10}$$

$$A_n \sim (n+1)(1-b)^{-1/2} - \frac{b}{2}(1-b)^{-3/2}, \qquad n \text{ large}$$

The term corresponding to $m = 0$ in (3.8–10) is $n + 1$.

We thus obtain

$$p(t, R) = \frac{e^{-a^2z^2}}{4b_0b_2^4}\frac{(Bz)^{3/2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}A_n$$

$$= \frac{e^{-a^2z^2}}{4\sqrt{\pi}}\frac{b_2^{1/2}}{b_0}(a^2-1)^{3/2}z^{3/2}\sum_{n=0}^{\infty}\frac{z^nA_n}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)} \tag{3.8–11}$$

We are interested in the expected number, N, of maxima per second. From the similar work for I, it follows that N is the coefficient of dt when (3.8–1) is integrated with respect to R from 0 to ∞. Thus from (3.8–7) and

$$dR = \sqrt{2B_4}\,b_2^{-2}\,dz = (2b_0B)^{1/2}b_2^{-3/2}\,dz$$

$$= [2b_0(a^2-1)]^{1/2}\,dz$$

we find

$$N = \int_0^{\infty} p(t, R)\,dR$$

$$= \frac{(a^2-1)^2}{(2a)^{5/2}}\left(\frac{b_2}{\pi b_0}\right)^{1/2}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}+\frac{5}{4}\right)}{\Gamma\left(\frac{n}{2}+\frac{7}{4}\right)}\frac{A_n}{a^n} \tag{3.8–12}$$

Equations (3.8–11) and (3.8–12) have been derived on the assumption that $w(f)$ is symmetrical about f_m, i.e. the band pass filter attenuation is

symmetrical about the mid-band frequency. We now go a step further and assume an ideal band pass filter:

$$\begin{aligned} w(f) &= w_0 \qquad f_a < f < f_b \\ w(f) &= 0 \qquad \text{otherwise} \\ 2f_m &= f_a + f_b \end{aligned} \tag{3.8–13}$$

Putting these in (3.8–2) we obtain zero for b_1 and b_3 and also

$$\begin{aligned} b_0 &= w_0(f_b - f_a) = \psi_0 \\ b_2 &= \frac{\pi^2 w_0}{3}(f_b - f_a)^3 \\ b_4 &= \frac{\pi^4 w_0}{5}(f_b - f_a)^5 \\ a^2 &= \tfrac{9}{5} \\ b &= \tfrac{1}{2}(3 - a^2) = \tfrac{3}{5} \\ R &= [2b_0(a^2 - 1)]^{1/2} z = [\tfrac{8}{5}\psi_0]^{1/2} z \\ \left(\frac{b_2}{\pi b_0}\right)^{1/2} &= \left[\frac{\pi}{3}\right]^{1/2}(f_b - f_a), \qquad a^2 z^2 = \frac{9R^2}{8\psi_0} \end{aligned} \tag{3.8–14}$$

n	A_n	n	A_n
0	1	4	6.775
1	2.3	5	8.333
2	3.735	6	9.9002
3	5.238	7	11.4736

$$A_n \sim 1.5811\, n + .3953$$

From (3.8–12) we find that the expected number of maxima per second of the envelope is

$$N = .64110\,(f_b - f_a) \tag{3.8–15}$$

assuming an ideal band pass filter.

The distribution of the maxima of R for an ideal band pass filter may be obtained by placing the results of (3.8–14) in (3.8–11). This gives

$$p(t, R)\,dR = \frac{dR}{\psi_0^{1/2}} \frac{(f_b - f_a)}{4} \sqrt{\frac{\pi}{3}} \left(\frac{4z}{5}\right)^{3/2} e^{-a^2 z^2} \sum_{n=0}^{\infty} \frac{z^n A_n}{\Gamma\left(\frac{n}{2} + \frac{7}{4}\right)}$$

It is convenient to define y as the ratio

$$y = \frac{R}{\text{r.m.s. } I(t)} = \frac{R}{\psi_0^{1/2}} = \left(\tfrac{8}{5}\right)^{1/2} z$$

where R is understood to correspond to a maximum of the envelope. Since the value of R corresponding to a maximum of the envelope selected at random is a random variable, y is also a random variable. Its probability density is $p_R(y)$, where

$$p_R(y)\, dy = \frac{p(t, R)\, dR}{0.64110(f_b - f_a)}$$

$p_R(y)$ has been computed and is plotted as a function of y in Fig. 3.

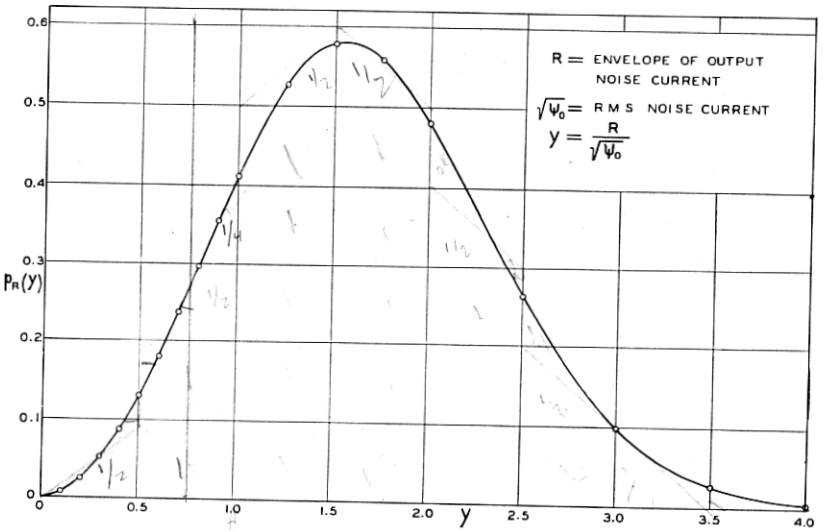

Fig. 3—Distribution of maxima of envelope of noise current. Noise through ideal band-pass filter.

$\frac{p_R(y)}{\sqrt{\psi_0}}\, dR$ = probability that a maximum of R selected at random lies between R and $R + dR$.

The distribution function $P(R_{\max} < y\sqrt{\psi_0})$ defined by

$$P(R_{\max} < y\sqrt{\psi_0}) = \int_0^y p_R(y)\, dy$$

and which gives the probability that a maximum of the envelope selected at random is less than a specified value $y\sqrt{\psi_0} = R$, is plotted in Fig. 4 together with other curves of the same nature.

When y is large, say greater than 2.5,

$$p_R(y) \sim \frac{\sqrt{\frac{\pi}{6}}}{.64110}(y^2 - 1)e^{-y^2/2}$$

$$P(R_{\max} < y\sqrt{\psi_0}) \sim 1 - \frac{\sqrt{\frac{\pi}{6}}}{.64110} ye^{-y^2/2}$$

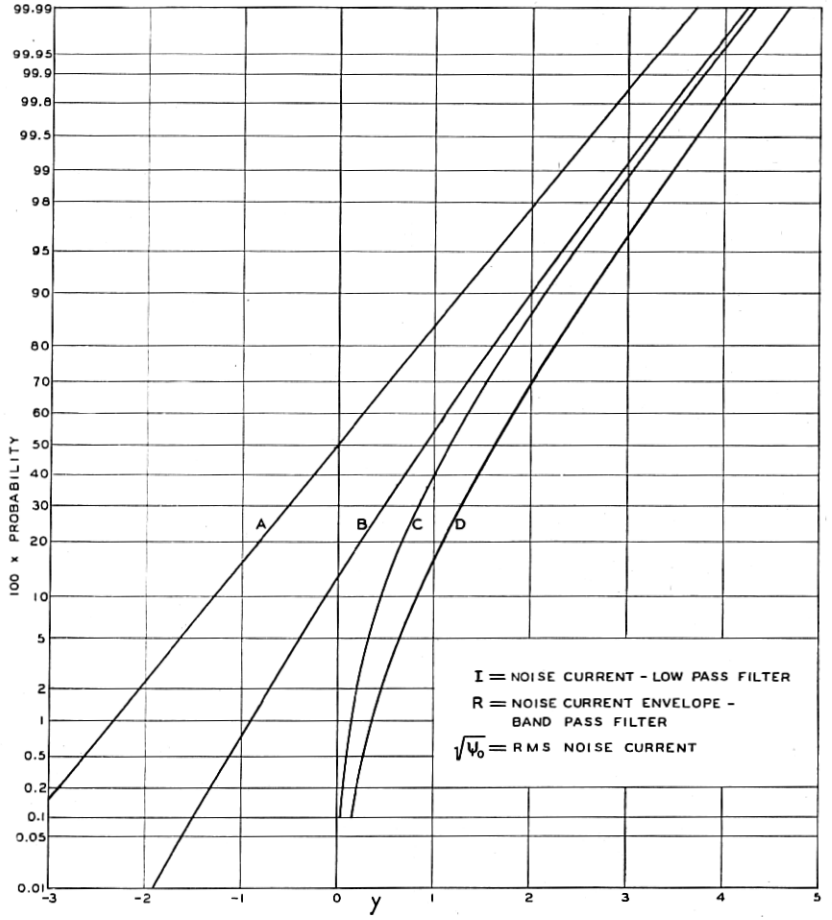

Fig. 4—Distribution of maxima

$A = P(I < y\sqrt{\psi_0})$ = probability of I being less than $y\sqrt{\psi_0}$. Similarly $C = P(R < y\sqrt{\psi_0})$.

$B = P(I \max < y\sqrt{\psi_0})$ = probability of random maximum of I being less than $y\sqrt{\psi_0}$. Similarly $D = P(R \max < y\sqrt{\psi_0})$.

The asymptotic expression for $p_R(y)$ may be obtained from the integral (3.8–9) for $p(t, R)$. Indeed, replacing the variables of integration x, y in (3.8–9) by

$$x' = x$$

$$y' = x + y,$$

integrating a portion of the y' integral by parts, and assuming $b < 1$ ($a^2 \geq 1$, by Schwarz's inequality, so that $b \leq 1$ always) leads to

$$p(t, R) \sim \left(\frac{b_2}{2\pi}\right)^{\frac{1}{2}} \frac{e^{-R^2/2\psi_0}}{\psi_0} \left(\frac{R^2}{\psi_0} - 1\right)$$

when R is large.

If, instead of an ideal band pass filter, we assume that $w(f)$ is given by

$$w(f) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(f-f_m)^2/2\sigma^2}, \qquad f_m \gg \sigma \tag{3.8–16}$$

we find that

$$b_0 = 1$$

$$b_2 = 4\pi^2\sigma^2$$

$$b_4 = 16\pi^4 \cdot 3\sigma^4$$

$$a^2 = 3, \; b = 0$$

$$A_n = (n + 1)$$

Some rough work indicates that the sum of the series in (3.8–12) is near 3.97. This gives the expected number of maxima of the envelope as

$$N = 2.52\sigma \tag{3.8–17}$$

per second.

The pass band is determined by σ. It appears difficult to compare this with an ideal band pass filter. If we use the fact that the filter given by

$$w(f) = w_0 \exp\left[-\pi\left(\frac{f - f_m}{f_b - f_a}\right)^2\right]$$

passes the same average amount of power as does an ideal band pass filter whose pass band is $f_b - f_a$, we have

$$f_b - f_a = \sigma\sqrt{2\pi}$$

and the expression for N becomes $1.006\,(f_b - f_a)$.

3.9 Energy Fluctuation

Some information regarding the statistical behavior of the random variable

$$E = \int_{t_1}^{t_1+T} I^2(t)\, dt \tag{3.9–1}$$

where $I(t)$ is a noise current and t_1 is chosen at random, has been given in a recent article.[35] Here we study this behavior from a somewhat different point of view.

If we agree to use the representations (2.8–1) or (2.8–6) we may write, as in the paper, the random variable E as

$$E = \int_{T/2}^{T/2} I^2(t)\, dt \tag{3.9–2}$$

where the randomness on the right is due either to the a_n's and b_n's if (2.8–1) is used or to the φ_n's if (2.8–6) is used.

The average value of E is m_T where, from (3.1–2),

$$\begin{aligned} \bar{E} = m_T &= \int_{T/2}^{T/2} \overline{I^2(t)}\, dt = \int_{T/2}^{T/2} \psi(0)\, dt = T\psi_0 \\ &= T\int_0^\infty w(f)\, df \end{aligned} \tag{3.9–3}$$

The second moment of E is

$$\overline{E^2} = \int_{T/2}^{T/2} dt_1 \int_{T/2}^{T/2} dt_2\, \overline{I^2(t_1) I^2(t_2)} \tag{3.9–4}$$

If, for the time being, we set t_2 equal to $t_1 + \tau$, it is seen from section 3.2 that we have an expression for the probability density of $I(t_1)$ and $I(t_1 + \tau)$ and hence we may obtain the required average:

$$\begin{aligned} \overline{I_1^2 I_2^2} = \frac{1}{2\pi A} \int_{-\infty}^{+\infty} dI_1 \int_{-\infty}^{+\infty} dI_2\, I_1^2 I_2^2 \exp \\ \left(-\frac{1}{2A^2}\left(\psi_0 I_1^2 + \psi_0 I_2^2 - 2\psi_\tau I_1 I_2\right)\right) \end{aligned} \tag{3.9–5}$$

$$A^2 = \psi_0^2 - \psi_\tau^2, \quad I_1 = I(t_1), \quad I_2 = I(t_1 + \tau) = I(t_2)$$

The integral may be evaluated by (3.5–6) when we set

$$\begin{aligned} I_1 &= Ax\sqrt{\frac{2}{\psi_0}}, \qquad I_2 = Ay\sqrt{\frac{2}{\psi_0}} \\ \psi_\tau &= -\psi_0 \cos\varphi \\ A &= \psi_0 \sin\varphi \end{aligned} \tag{3.9–6}$$

[35] "Filtered Thermal Noise—Fluctuation of Energy as a Function of Interval Length", *Jour. Acous. Soc. Am.*, 14 (1943), 216–227.

Thus

$$\begin{aligned}\overline{I_1^2 I_2^2} &= \psi_0^2(1 + 2\cos^2\varphi) \\ &= \psi_0^2 + 2\psi_\tau^2\end{aligned} \tag{3.9-7}$$

Incidentally, this gives an expression for the correlation function of $I^2(t)$. Replacing τ by its value of $t_2 - t_1$ and returning to (3.9-4),

$$\overline{E^2} = T^2\psi_0^2 + 2\int_{T/2}^{T/2} dt_1 \int_{T/2}^{T/2} dt_2 \psi^2(t_2 - t_1) \tag{3.9-8}$$

When we introduce σ_T, the standard deviation of E, and use

$$\sigma_T^2 = \overline{E^2} - m_T^2$$

we obtain

$$\begin{aligned}\sigma_T^2 = \overline{(E - \bar{E})^2} &= 2\int_{T/2}^{T/2} dt_1 \int_{T/2}^{T/2} dt_2 \psi^2(t_2 - t_1) \\ &= 4\int_0^T (T - x)\psi^2(x)\, dx\end{aligned}$$

where the second line may be obtained from the first either by changing the variables of integration, as in (3.9-27), or by the method used below in dealing with $\overline{E^3}$. I am indebted to Prof. Kac for pointing out the advantage obtained by reducing the double integral to a single integral. It should be noted that the limits of integration $-T/2$, $T/2$ in the double integral may be replaced by 0, T by making the change of variable $t = t' - T/2$ for both t_1 and t_2.

When we use

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau\, df \tag{2.1-6}$$

we obtain the result stated in the paper, namely,

$$\sigma_T^2 = \int_0^\infty w(f_1)\, df_1 \int_0^\infty w(f_2)\, df_2 \left[\frac{\sin^2 \pi(f_1 + f_2)T}{\pi^2(f_1 + f_2)^2} + \frac{\sin^2 \pi(f_1 - f_2)T}{\pi^2(f_1 - f_2)^2}\right] \tag{3.9-9}$$

If this formula is applied to a relatively narrow band-pass filter and if $T(f_b - f_a) >> 1$ the contribution of the $f_1 + f_2$ term may be neglected and we have the approximation

$$\begin{aligned}\sigma_T^2 &\approx \int_{f_a}^{f_b} w_0\, df_1 \int_{-\infty}^{+\infty} w_0\, df_2 \frac{\sin^2 \pi(f_1 - f_2)T}{\pi^2(f_1 - f_2)^2} \\ &= w_0^2 T(f_b - f_a) \\ &= w_0 m_T\end{aligned} \tag{3.9-10}$$

where, from (3.9–3)

$$m_T = w_0 T(f_b - f_a) \tag{3.9–11}$$

The third moment $\overline{E^3}$ may be computed in the same way. However, in this case it pays to introduce the characteristic function for the distribution of $I(t_1)$, $I(t_2)$, $I(t_3)$. Since this distribution is normal its characteristic function is

$$\begin{aligned}\text{Average exp } [iz_1 I_1 + iz_2 I_2 + iz_3 I_3] \\ = \exp -\Big[\frac{\psi_0}{2}(z_1^2 + z_2^2 + z_3^2) + \psi(t_2 - t_1)z_1 z_2 \\ + \psi(t_3 - t_1)z_1 z_3 + \psi(t_3 - t_2)z_2 z_3\Big]\end{aligned} \tag{3.9–12}$$

From the definition of the characteristic function it follows that

$$\begin{aligned}\overline{I_1^2 I_2^2 I_3^2} &= -\text{coeff. of } \frac{z_1^2 z_2^2 z_3^2}{2!2!2!} \text{ in ch. f.} \\ &= \psi_0^3 + 2\psi_0(\psi_{21}^2 + \psi_{31}^2 + \psi_{32}^2) \\ &\quad + 8\psi_{21}\psi_{31}\psi_{32}\end{aligned} \tag{3.9–13}$$

where we have written ψ_{21} for $\psi(t_2 - t_1)$, etc. When (3.9–13) is multiplied by $dt_1\, dt_2\, dt_3$, the variables integrated from 0 to T, and the above double integral expression for σ_T^2 used, we find

$$\overline{(E - \bar{E})^3} = 2!2^2 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \psi_{21}\psi_{31}\psi_{32}\,.$$

Denoting the triple integral on the right by J and differentiating,

$$\begin{aligned}\frac{dJ}{dT} &= 3\int_0^T dt_1 \int_0^T dt_2 \psi(t_2 - t_1)\psi(T - t_1)\psi(T - t_2) \\ &= 3\int_0^T dx \int_0^T dy \psi(x - y)\psi(x)\psi(y) \\ &= 6\int_0^T dx \int_0^x dy \psi(x - y)\psi(x)\psi(y)\end{aligned}$$

In going from the first line to the second t_1 and t_2 were replaced by $T - x$ and $T - y$, respectively. In going from the second to the third use was made of the relations symbolized by

$$\begin{aligned}\int_0^T dx \int_0^T dy &= \int_0^T dx \int_0^x dy + \int_0^T dx \int_x^T dy \\ &= \int_0^T dx \int_0^x dy + \int_0^T dy \int_0^y dx\end{aligned}$$

and of the fact that the integrand is symmetrical in x and y. Integrating dJ/dT with respect to T from 0 to T_1, using the formula

$$\int_0^{T_1} dT \int_0^T f(x)\,dx = \int_0^{T_1} (T_1 - x)f(x)\,dx,$$

noting that J is zero when T is zero, and dropping the subscript on T_1 finally gives

$$\overline{(E - \bar{E})^3} = 48 \int_0^T dx \int_0^x dy(T - x)\psi(x)\psi(y)\psi(x - y).$$

$\overline{E^4}$ may be treated in a similar way. It is found that

$$\overline{(E - \bar{E})^4} - 3\overline{(E - \bar{E})^2}^2 = 3!2^3 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \int_0^T dt_4 \psi_{21}\psi_{31}\psi_{42}\psi_{43}$$

which may be reduced to the sum of two triple integrals. It is interesting to note that the expression on the left is the fourth semi-invariant of the random variable E and gives us a measure of the peakedness of the distribution (kurtosis). Likewise, the second and third moments about the mean are the second and third semi-invariants of E. This suggests that possibly the higher semi-invariants may also be expressed as similar multiple integrals.

So far, in this section, we have been speaking of the statistical constants of E. The determination of an exact expression for the probability density of E, in which T occurs as a parameter, seems to be quite difficult.

When T is very small E is approximately $I^2(t)T$. The probability that E lies in dE is the probability that the current lies in $-I, -I - dI$ plus the probability that the current lies in $I, I + dI$:

$$\frac{2dI}{\sqrt{2\pi\psi_0}} \exp - \frac{I^2}{2\psi_0} = (2\pi\psi_0 ET)^{-1/2} \exp - \frac{E}{2\psi_0 T}\,dE \qquad (3.9\text{-}14)$$

where E is positive,

$$I = \left(\frac{E}{T}\right)^{1/2}, \qquad dI = \frac{1}{2}(ET)^{-1/2}\,dE$$

and T is assumed to be so small that $I(t)$ does not change appreciably during an interval of length T.

When T is very large we may divide it into a number of intervals, say n, each of length T/n. Let E_r be the contribution of the rth interval. The energy E for the entire interval is then

$$E = E_1 + E_2 + \cdots + E_n$$

If the sub-intervals are large enough the E_r's are substantially independent random variables. If in addition n is large enough E is distributed nor-

mally, approximately. Hence when T is very large the probability that E lies in dE is

$$\frac{dE}{\sigma_T\sqrt{2\pi}} \exp - \frac{(E - m_T)^2}{2\sigma_T^2} \tag{3.9–15}$$

where

$$\begin{aligned} m_T &= T \int_0^\infty w(f)\, df \\ \sigma_T^2 &= T \int_0^\infty w^2(f)\, df \end{aligned} \tag{3.9–16}$$

the second relation being obtained by letting $T \to \infty$ in (3.9–9). The analogy with Campbell's theorem, section 1.2, is evident. When we deal with a band pass filter we may use (3.9–10) and (3.9–11).

Consider a relatively narrow band pass filter such that we may find a T for which $Tf_a >> 2\pi$ but $T(f_b - f_a) << .64$. Thus several cycles of frequency f_a are contained in T but, from (3.8–15), the envelope does not change appreciably during this interval. Thus throughout this interval $I(t)$ may be considered to be a sine wave of amplitude R. The corresponding value of E is approximately

$$E = T\frac{R^2}{2}$$

where the distribution of the envelope R is given by (3.7–10). From this it follows that the probability of E lying in dE is

$$\frac{dE}{\psi_0 T} \exp - \frac{E}{\psi_0 T} = \frac{dE}{m_T} e^{-E/m_T} \tag{3.9–17}$$

when E is small but not too small.

When we look at (3.9–14) and (3.9–17) we observe that they are of the form

$$\frac{a^{n+1}E^n}{\Gamma(n+1)} e^{-aE}\, dE \tag{3.9–18}$$

Moreover, the normal law (3.9–15), may be obtained from this by letting n become large. This suggests that an approximate expression for the distribution of E is given by (3.9–18) when a and n are selected so as to give the values of m_T and σ_T obtained from (3.9–3) and (3.9–9). This gives

$$a = \frac{m_T}{\sigma_T^2}, \qquad n + 1 = \frac{m_T^2}{\sigma_T^2} \tag{3.9–19}$$

and if we drop the subscript T and substitute the value of a in (3.9–18) we get

$$\frac{\left(\frac{mE}{\sigma^2}\right)^n}{\Gamma(n+1)} \exp\left(-\frac{mE}{\sigma^2}\right) d\left(\frac{mE}{\sigma^2}\right), \qquad n = \frac{m^2}{\sigma^2} - 1 \qquad (3.9\text{–}20)$$

An idea of how this distribution behaves may be obtained from the following table:

n	$T(f_b - f_a)$	$x_{.25}$	$x_{.50}$	$x_{.75}$	$\frac{x_{.25}}{x_{.50}}$	$\frac{x_{.75}}{x_{.50}}$
0	0	.29	.695	1.39	.415	2.00
1	1.45	.96	1.68	2.69	.572	1.60
2	2.4	1.73	2.67	3.94	.647	1.47
3	3.4	2.54	3.67	5.12	.692	1.39
5	5.4	4.22	5.67	7.42	.744	1.31
10	10.5	8.63	10.67	13.02	.808	1.22
24	25	21.47	24.67	28.17	.870	1.14
48	50	44.1	48.7	53.5	.905	1.10

where n is the exponent in (3.9–20). The column $T(f_b - f_a)$ holds only for a narrow band pass filter and was obtained by reading the curve y_A in Fig. 1 of the above mentioned paper. The figures in this column are not very accurate. The next three columns give the points which divide the distribution into four intervals of equal probability:

$$x_{.25} = \frac{mE_{.25}}{\sigma^2}, \quad E_{.25} = \text{energy exceeded 75\% of time}$$

$$x_{.50} = \frac{mE_{.50}}{\sigma^2}, \quad E_{.50} = \text{energy exceeded 50\% of time}$$

$$x_{.75} = \frac{mE_{.75}}{\sigma^2}, \quad E_{.75} = \text{energy exceeded 25\% of time}$$

The values in these columns were obtained from Pearson's table of the incomplete gamma function. The last two columns show how the distribution clusters around the average value as the normal law is approached.

For the larger values of n we expected the normal law (3.9–15) to be approached. Since, for this law the 25, 50, and 75 per cent points are at $m - .675\sigma$, m, and $m + .675\sigma$ we have to a first approximation

$$\begin{aligned} x_{.50} &= \frac{m^2}{\sigma^2} = (n+1) \approx T(f_b - f_a) \\ x_{.25} &= \frac{m}{\sigma^2}(m - .675\sigma) = x_{.50} - .675\sqrt{x_{.50}} \\ x_{.75} &= x_{.50} + .675\sqrt{x_{.50}} \end{aligned} \qquad (3.9\text{–}21)$$

This agrees with the table.

Thiede[36] has studied the mean square value of the fluctuations of the integral

$$A(t) = \int_{-\infty}^{t} I^2(\tau) e^{-\alpha(t-\tau)}\, d\tau \qquad (3.9\text{–}22)$$

The reading of a hot wire ammeter through which a current I is passing is proportional to $A(t)$. α is a constant of the meter. Here we study $A(t)$ by

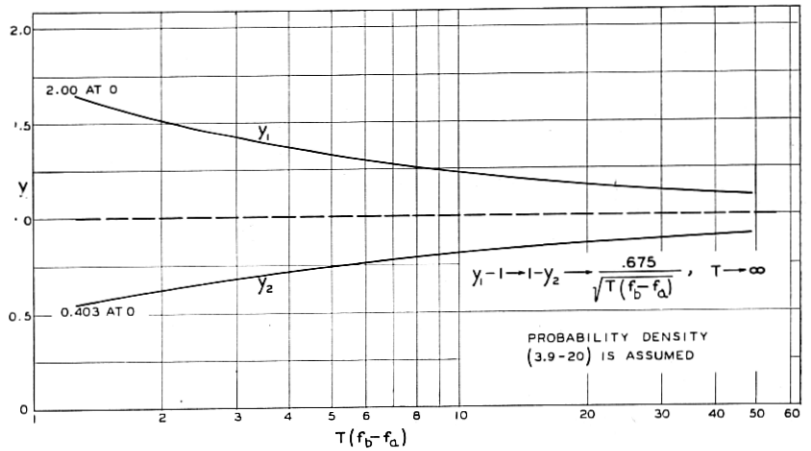

Fig. 5*—Filtered thermal noise—spread of energy fluctuation

$$E = \int_{t_1}^{t_1+T} I^2(t)\, dt, \qquad t_1 \text{ random}, \qquad I \text{ is noise current.}$$

$y_1 = E_{.75}/E_{.50}$, $y_2 = E_{.25}/E_{.50}$.
$f_b - f_a$ = band width of filter.

first obtaining its correlation function. This method of approach enables us to extend Thiede's results

The distributed portion of the power spectrum of $A(t)$ is given by (3.9–30). When the power spectrum $w(f)$ of $I(t)$ is zero except over the band $f_a < f < f_b$ where it is w_0, the power spectrum of $A(t)$ is

$$\frac{2w_0^2(f_b - f_a - f)}{\alpha^2 + 4\pi^2 f^2} \quad \text{for} \quad 0 < f < f_b - f_a$$

and is zero from $f_b - f_a$ up to $2f_a$. The spectrum from $2f_a$ to $2f_b$ is not zero, and may be obtained from (3.9–34). The mean square fluctuation of $A(t)$ is given, in the general case, by (3.9–28) and (3.9–32). For the band pass case, when $(f_b - f_a)/\alpha$ is large,

$$\text{r.m.s.}\ \frac{A(t) - \bar{A}}{\bar{A}} = \left[\frac{\alpha}{2(f_b - f_a)}\right]^{1/2}$$

[36] *Elec. Nachr. Tek.*, 13 (1936), 84–95. This is an excellent article.
* Note added in proof. The value of y_2 at 0 should be .415 instead of .403.

We start by setting $\tau = t - u$ which transforms the integral for $A(t)$ into

$$A(t) = \int_0^\infty I^2(t - u)e^{-\alpha u}\, du \tag{3.9–23}$$

In order to obtain the correlation function $\Psi(\tau)$ for $A(t)$ we multiply $A(t)$ by $A(t + \tau)$ and average over all the possible currents

$$\begin{aligned}\Psi(\tau) &= \overline{A(t)A(t + \tau)} \\ &= \int_0^\infty e^{-\alpha u}\, du \int_0^\infty e^{-\alpha v}\, dv \text{ ave. } I^2(t - u)I^2(t + \tau - v)\end{aligned}$$

Just as in (3.9–4) the average in the integrand is the correlation function of $I^2(t)$, the argument being $t + \tau - v - t + u = \tau + u - v$. From (3.9–7) it is seen that this is

$$\psi_0^2 + 2\psi^2(\tau + u - v)$$

where $\psi(\tau)$ is the correlation function of $I(t)$. Hence

$$\Psi(\tau) = \frac{\psi_0^2}{\alpha^2} + 2\int_0^\infty du \int_0^\infty dv\, e^{-\alpha u - \alpha v}\psi^2(\tau + u - v) \tag{3.9–24}$$

From the integral (3.9–23) for $A(t)$ it is seen that the average value of $A(t)$ is

$$\bar{A} = \frac{\bar{I}^2}{\alpha} = \frac{\psi_0}{\alpha} \tag{3.9–25}$$

where we have used

$$\psi_0 = \psi(0) = \int_0^\infty w(f)\, df = \bar{I}^2$$

Using this result again, only this time applying it to $A(t)$, gives

$$\begin{aligned}\overline{A^2(t)} &= \Psi(0) \\ &= \bar{A}^2 + 2\int_0^\infty du \int_0^\infty dv\, e^{-\alpha u - \alpha v}\psi^2(u - v)\end{aligned} \tag{3.9–26}$$

The double integrals may be transformed by means of the change of variable $u + v = x$, $u - v = y$. Then (3.9–24) becomes

$$\begin{aligned}\Psi(\tau) &= \bar{A}^2 + \left[\int_0^\infty dy \int_y^\infty dx + \int_{-\infty}^0 dy \int_{-y}^\infty dx\right] e^{-\alpha x}\psi^2(\tau + y) \\ &= \bar{A}^2 + \frac{1}{\alpha}\int_0^\infty e^{-\alpha y}[\psi^2(\tau + y) + \psi^2(\tau - y)]\, dy\end{aligned} \tag{3.9–27}$$

When we make use of the fact that $\psi(y)$ is an even function of y we see, from (3.9–26), that the mean square fluctuation of $A(t)$ is

$$\overline{(A(t) - \bar{A})^2} = \overline{A^2(t)} - \bar{A}^2 = \frac{2}{\alpha}\int_0^\infty e^{-\alpha y}\psi^2(y)\,dy \qquad (3.9\text{–}28)$$

$\Psi(\tau)$ may be expressed in terms of integrals involving the power spectrum $w(f)$ of $I(t)$. The work starts with (3.9–24) and is much the same as in going from (3.9–8) to (3.9–9). The result is

$$\Psi(\tau) = \bar{A}^2 + \int_0^\infty df_1 \int_0^\infty df_2 w(f_1)w(f_2) \left[\frac{\cos 2\pi(f_1 + f_2)\tau}{\alpha^2 + [2\pi(f_1 + f_2)]^2} + \frac{\cos 2\pi(f_1 - f_2)\tau}{\alpha^2 + [2\pi(f_1 - f_2)]^2}\right]$$

It is convenient to define $w(-f)$ for negative frequencies to be equal to $w(f)$. The integration with respect to f_2 may then be taken from $-\infty$ to $+\infty$ and we get

$$\Psi(\tau) = \bar{A}^2 + \int_0^\infty df_1 \int_{-\infty}^{+\infty} df_2 w(f_1)w(f_2) \frac{\cos 2\pi(f_1 - f_2)\tau}{\alpha^2 + [2\pi(f_1 - f_2)]^2} \qquad (3.9\text{–}29)$$

The power spectrum $W(f)$ of $A(t)$ may be obtained by integrating $\Psi(\tau)$:

$$W(f) = 4\int_0^\infty \Psi(\tau)\cos 2\pi f\tau\,d\tau$$

Let us concern ourselves with the fluctuating portion $A(t) - \bar{A}$ of $A(t)$. Its power spectrum $W_c(f)$ is

$$W_c(f) = 4\int_0^\infty (\Psi(\tau) - \bar{A}^2)\cos 2\pi f\tau\,d\tau$$

The integration is simplified by using Fourier's integral formula in the form

$$\int_0^\infty d\tau \int_{-\infty}^{+\infty} df_2 F(f_2)\cos 2\pi(u - f_2)\tau = \tfrac{1}{2}F(u)$$

We get

$$W_c(f) = \frac{1}{\alpha^2 + 4\pi^2 f^2}\int_0^\infty df_1[w(f_1)w(f + f_1) + w(f_1)w(-f + f_1)] = \frac{1}{\alpha^2 + 4\pi^2 f^2}\int_{-\infty}^{+\infty} w(f_1)w(f - f_1)\,df_1 \qquad (3.9\text{–}30)$$

The simplicity of this result suggests that a simpler derivation may be found. If we attempt to use the result

$$\bar{w}(f) = \operatorname*{Limit}_{T\to\infty} \frac{\overline{2|S(f)|^2}}{T} \qquad (2.5\text{–}3)$$

where $S(f)$ is given by (2.1–2) we find that we need the result

$$\underset{T\to\infty}{\text{Limit}}\ \frac{2}{T}\int_0^T dt_1 \int_0^T dt_2\, e^{2\pi i f(t_2-t_1)}\, I^2(t_1)I^2(t_2) = \int_{-\infty}^{+\infty} w(f_1)w(f-f_1)\,df_1 \tag{3.9–31}$$

where $f > 0$ and $I(t)$ is a noise current with $w(f)$ as its power spectrum. This may be proved by using (3.9–7) and

$$8\int_0^\infty \psi^2(\tau)\cos 2\pi f\tau\, d\tau = \int_{-\infty}^{+\infty} w(x)w(f-x)\,dx$$

which is given by equation (4C-6) in Appendix 4C.

An expression for the mean square fluctuation of $A(t)$ in terms of $w(f)$ may be obtained by setting τ equal to zero in (3.9–29)

$$\overline{(A(t)-\bar{A})^2} = \Psi(0) - \bar{A}^2 = \int_0^\infty df_1 \int_{-\infty}^{+\infty} df_2 \frac{w(f_1)w(f_2)}{\alpha^2 + 4\pi^2(f_1-f_2)^2} \tag{3.9–32}$$

The same result may be obtained by integrating $W_c(f)$, (3.9–30), from 0 to ∞:

$$\int_0^\infty \frac{df}{\alpha^2+4\pi^2f^2}\int_{-\infty}^{+\infty} df_1\, w(f_1)w(f-f_1) \tag{3.9–33}$$

Although this differs in appearance from (3.9–32) it may be transformed into that expression by making use of $w(-f) = w(f)$.

Suppose that $I(t)$ is the current through an ideal band pass filter so that $w(f)$ is zero except in the band $f_a < f < f_b$ where it is w_0. Then, if $3f_a > f_b$,

$$\bar{A} = \frac{w_0}{\alpha}(f_b - f_a) \tag{3.9–34}$$

$$\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx = \begin{cases} 2w_0^2(f_b - f_a - f) & 0 < f \le f_b - f_a \\ w_0^2(f - 2f_a) & 2f_a \le f \le f_b + f_a \\ w_0^2(2f_b - f) & f_b + f_a \le f \le 2f_b \end{cases}$$

and is zero outside these ranges. The power spectrum $W_c(f)$ may be obtained immediately from (3.9–30) by dividing these values by $\alpha^2 + 4\pi^2f^2$.

From (3.9–33)

$$\overline{(A(t)-\bar{A})^2} = 2w_0^2\int_0^{f_b-f_a}\frac{(f_b-f_a-f)\,df}{\alpha^2+4\pi^2f^2} + w_0^2\int_{2f_a}^{f_b+f_a}\frac{(f-2f_a)}{\alpha^2+4\pi^2f^2}\,df + w_0^2\int_{f_b+f_a}^{2f_b}\frac{(2f_b-f)}{\alpha^2+4\pi^2f^2}\,df$$

If an exact answer is desired the integrations may be performed. When we assume that $f_b - f_a << f_b + f_a$ we may obtain approximations for the last two integrals.

$$\overline{(A(t) - \bar{A})^2} \approx w_0^2 \left[\frac{f_b - f_a}{\pi\alpha} \tan^{-1} \frac{2\pi(f_b - f_a)}{\alpha} - \frac{1}{4\pi^2} \log \frac{\alpha^2 + 4\pi^2(f_b - f_a)^2}{\alpha^2} + \frac{(f_b - f_a)^2}{\alpha^2 + 4\pi^2(f_b + f_a)^2} \right]$$

Furthermore, if $2\pi(f_b - f_a)/\alpha$ is large we have

$$\overline{(A(t) - \bar{A})^2} \approx w_0^2 \frac{f_b - f_a}{2\alpha}$$

and the relative r.m.s. fluctuation is

$$\text{r.m.s. of } \left[\frac{(A(t) - \bar{A})}{\bar{A}} \right] \approx \left[\frac{\alpha}{2(f_b - f_a)} \right]^{1/2}$$

This result may also be obtained from (3.9–10) and (3.9–11) by assuming α so small that the integral for $A(t)$ may be broken into a great many integrals each extending over an interval T. αT is assumed so small that $e^{-\alpha u}$ is substantially constant over each interval.

3.10 Distribution of Noise Plus Sine Wave

Suppose we have a steady sinusoidal current

$$I_p = I_p(t) = P \cos(\omega_p t - \varphi_p) \tag{3.10–1}$$

We pick times $t_1, t_2, \cdots$ at random and note the corresponding values of the current. How are these values distributed? Picking the times at random in (3.10–1) is the same, statistically, as holding t constant and picking the phase angles φ_p at random from the range 0 to 2π. If I_p be regarded as a random variable defined by the random variable φ_p, its characteristic function is

$$\begin{aligned} \text{ave. } e^{izI_p} &= \frac{1}{2\pi} \int_0^{2\pi} e^{izP \cos(\omega_p t - \varphi)} \, d\varphi \\ &= J_0(Pz) \end{aligned} \tag{3.10–2}$$

and its probability density is

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-izI_p} J_0(Pz) \, dz = \begin{cases} \frac{1}{\pi}(P^2 - I_p^2)^{-1/2} & |I_p| < P \\ 0 & |I_p| > P \end{cases} \tag{3.10–3}$$

In this case it is simpler to obtain the probability density directly from (3.10–1) instead of from the characteristic function.

Now suppose that we have a noise current I_N plus a sine wave. By combining our representation (2.8–6) for I_N with the idea of φ_p being random mentioned above we are led to the representation

$$\begin{aligned} I(t) &= I = I_p + I_N \\ &= P\cos(\omega_p t - \varphi_p) + \sum_1^M c_n \cos(\omega_n t - \varphi_n), \qquad (3.10\text{–}4) \\ c_n^2 &= 2w(f_n)\Delta f \end{aligned}$$

where φ_p and $\varphi_1, \cdots \varphi_M$ are independent random angles.

If we note I at the random times $t_1, t_2 \cdots$ how are the observed values distributed? Since I_p and I_N may be regarded as independent random variables and since the characteristic function for the sum of two such variables is the product of their characteristic functions we have from (3.1–6) and (3.10–2)

$$\begin{aligned} \text{ave. } e^{izI} &= \text{ave. } e^{iz(I_p+I_N)} \\ &= J_0(Pz)\exp\left(\frac{-\psi_0 z^2}{2}\right) \qquad (3.10\text{–}5) \end{aligned}$$

which gives the characteristic function of I. The probability density of I is[37]

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-izI-(\psi_0 z^2/2)} J_0(Pz)\, dz = \frac{1}{\pi\sqrt{2\pi\psi_0}}\int_0^{\pi} e^{-(I-P\cos\theta)^2/2\psi_0}\, d\theta \qquad (3.10\text{–}6)$$

In the same way the two-dimensional probability density of (I_1, I_2), where $I_1 = I(t)$ is a sine wave plus noise (3.10–4) and $I_2 = I(t+\tau)$ is its value at a constant interval τ later, may be shown to be

$$\frac{(\psi_0^2 - \psi_\tau^2)^{-1/2}}{2\pi}\int_0^{2\pi} d\theta \exp\left[-\frac{B(\theta)}{2(\psi_0^2 - \psi_\tau^2)}\right] \qquad (3.10\text{–}7)$$

where

$$\begin{aligned} B(\theta) = \psi_0[(I_1 - P\cos\theta)^2 + (I_2 - P\cos(\theta + \omega_p\tau))^2] \\ - 2\psi_\tau(I_1 - P\cos\theta)(I_2 - P\cos(\theta + \omega_p\tau)) \end{aligned}$$

The characteristic function for I_1 and I_2 is

$$\begin{aligned} \text{ave. } e^{iuI_1+ivI_2} &= J_0\left(P\sqrt{u^2 + v^2 + 2uv\cos\omega_p\tau}\right) \\ &\times \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \qquad (3.10\text{–}8) \end{aligned}$$

[37] A different derivation of this expression is given by W. R. Bennett, *Jour. Acous. Soc. Amer.*, Vol. 15, p. 165 (Jan. 1944); *B.S.T.J.*, Vol. 23, p. 97 (Jan. 1944).

Sometimes the distribution of the envelope of

$$I = P \cos pt + I_N \tag{3.10–9}$$

is of interest. Here we have replaced ω_p by p and have set φ_p to zero. By the envelope we mean $R(t)$ given by

$$R^2(t) = R^2 = (P + I_c)^2 + I_s^2 \tag{3.10–10}$$

where I_c is the component of I_N "in phase" with cos pt and I_s is the component "in phase" with sin pt:

$$I_c = \sum c_n \cos [(\omega_n - p)t - \varphi_n]$$

$$I_s = \sum c_n \sin [(\omega_n - p)t - \varphi_n]$$

$$I_N = I_c \cos pt - I_s \sin pt$$

$$\overline{I_N^2} = \overline{I_c^2} = \overline{I_s^2} = \psi_0$$

Since I_c and I_s are distributed normally about zero with a variance of ψ_0, the probability densities of the variables

$$x = P + I_c$$

$$y = I_s$$

are

$$(2\pi\psi_0)^{-1/2} \exp - \frac{(x - P)^2}{2\psi_0}$$

$$(2\pi\psi_0)^{-1/2} \exp - \frac{y^2}{2\psi_0}$$

respectively. Setting

$$x = R \cos \theta$$

$$y = R \sin \theta$$

and using these distributions shows that the probability of a point (x, y) lying in the ring R, $R + dR$ is

$$\frac{R\,dR}{2\pi\psi_0} \int_0^{2\pi} \exp \left[-\frac{1}{2\psi_0} (R^2 + P^2 - 2RP \cos \theta) \right] d\theta$$

$$= \frac{R\,dR}{\psi_0} \exp \left[-\frac{R^2 + P^2}{2\psi_0} \right] I_0 \left(\frac{RP}{\psi_0} \right) \tag{3.10–11}$$

where I_0 is the Bessel function with imaginary argument.

$$I_0(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n} n!\, n!}$$

and is a tabulated function. Thus (3.10–11) gives the probability density of the envelope R.

The average value of R^n may be obtained by multiplying (3.10–11) by R^n and integrating from 0 to ∞. Expansion of the Bessel function and term-wise integration gives

$$\overline{R^n} = (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right)e^{-P^2/2\psi_0}\,{}_1F_1\left(\frac{n}{2}+1;\,1;\,\frac{P^2}{2\psi_0}\right)$$

$$= (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right){}_1F_1\left(-\frac{n}{2};\,1;\,-\frac{P^2}{2\psi_0}\right) \qquad (3.10\text{–}12)$$

where ${}_1F_1$ is a hypergeometric function.[38] In going from the first line to the second we have used Kummer's first transformation of this function. A special case is

$$\overline{R^2} = P^2 + 2\psi_0 \qquad (3.10\text{–}13)$$

When only noise is present, $P = 0$ and

$$\overline{R} = (2\psi_0)^{1/2}\Gamma(\tfrac{3}{2}) = \left(\frac{\psi_0\pi}{2}\right)^{1/2}$$
$$\overline{R^2} = 2\psi_0 \qquad (3.10\text{–}14)$$

Before going further with (3.10–11) it is convenient to make the following change of notation

$$v = \frac{R}{\psi_0^{1/2}}, \qquad dv = \frac{dR}{\psi_0^{1/2}}, \qquad a = \frac{P}{\psi_0^{1/2}} \qquad (3.10\text{–}15)$$

"a" is the ratio (sine wave amplitude)/(r.m.s. noise current).
Instead of the random variable R we now have the random variable v whose probability density is

$$p(v) = v\exp\left[-\frac{v^2+a^2}{2}\right]I_0(av) \qquad (3.10\text{–}16)$$

Curves of $p(v)$ versus v are plotted in Fig. 6 for the values 0, 1, 2, 3, 5 of a. Curves showing the probability that v is less than a stated amount, i.e., distribution curves for v, are given in Fig. 7. These curves were obtained by integrating $p(v)$ numerically. The following useful expression for this probability has been given by W. R. Bennett in some unpublished work.

$$\int_0^v p(u)\,du = \exp\left[-\frac{v^2+a^2}{2}\right]\sum_{n=1}^{\infty}\left(\frac{v}{a}\right)^n I_n(av) \qquad (3.10\text{–}17)$$

[38] Curves of this function are given in "Tables of Functions", Jahnke and Emde (1938), p. 275, and some of its properties are stated in Appendix 4C.

This is obtained by integration by parts using

$$\int u^n I_{n-1}(au)\, du = u^n I_n(au)/a$$

When $av >> 1$ but $1 << a - v$, Bennett has shown that (3.10–17) leads to

$$\int_0^v p(u)\, du \approx \left(\frac{v}{2\pi a}\right)^{1/2} \frac{1}{a - v} \exp\left[-\frac{(v - a)^2}{2}\right] \left(1 - \frac{3(a + v)^2 - 4v^2}{8av(a - v)^2} \cdots\right) \tag{3.10–18}$$

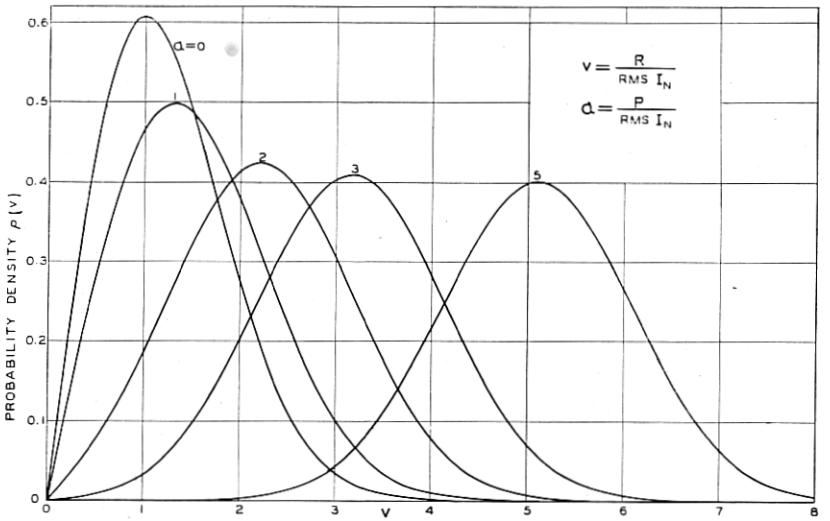

Fig. 6—Probability density of envelope R of $I(t) = P \cos pt + I_N$

This formula may also be obtained by putting the asymptotic expansion (3.10–19) for $p(v)$ in (3.10–17), integrating by parts twice, and neglecting higher order terms.

When av becomes large we may replace $I_0(av)$ by its asymptotic expression. The expression for $p(v)$ is then

$$p(v) \sim \left(1 + \frac{1}{8av}\right)\left(\frac{v}{2\pi a}\right)^{1/2} \exp\left[-\frac{(v - a)^2}{2}\right] \tag{3.10–19}$$

Thus when either a becomes large or v is far out on the tail of the probability density curve, the distribution behaves like a normal law. In terms of the original quantities, the normal law has an average of P and a standard deviation of $\psi_0^{1/2}$. This standard deviation is the same as the standard deviation

of the instantaneous values of I_N. When $av \gg 1$ and $a \gg |v - a|$ we may expand the coefficient of the exponential term in (3.10–19) in powers of

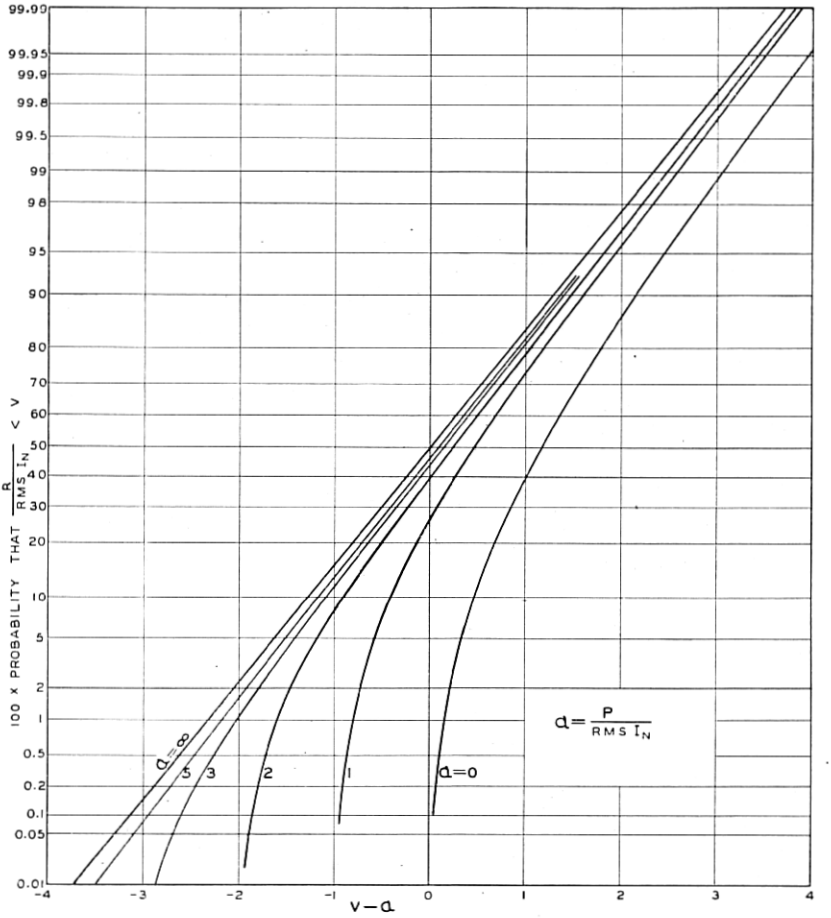

Fig. 7—Distribution function of envelope R of $I(t) = P \cos pt + I_N$

$(v - a)/a$. Integrating this expansion termwise gives, when terms of magnitude less than a^{-3} are neglected,

$$\int_0^v p(u)\,du \approx \frac{1}{2} + \frac{1}{2}\,erf\,\frac{v-a}{\sqrt{2}} - \frac{1}{2a\sqrt{2\pi}}\left[1 - \frac{v-a}{4a} + \frac{1+(v-a)^2}{8a^2}\right]\exp\left[-\frac{(v-a)^2}{2}\right]$$

When I consists of two sine waves plus noise

$$I = P \cos pt + Q \sin qt + I_N , \tag{3.10–20}$$

where the radian frequencies p and q are incommensurable, the probability density of the envelope R is

$$R \int_0^\infty rJ_0(Rr)J_0(Pr)J_0(Qr)e^{-\psi_0 r^2/2}\, dr \tag{3.10–21}$$

where ψ_0 is $\overline{I_N^2}$. When Q is zero the integral may be evaluated to give (3.10–11). When both P and Q are zero the probability density for R when only noise is present is obtained. If there are three sine waves instead of two then another Bessel function must be placed in the integrand, and so on. To define R it is convenient to think of the noise as being confined to a relatively narrow band and the frequencies of the sine waves lying within, or close to, this band. As in equations (3.7–2) to (3.7–4), we refer all terms to a representative mid-band frequency $f_m = \omega_m/2\pi$ by using equations of the type

$$\begin{aligned} \cos pt &= \cos [(p - \omega_m)t + \omega_m t] \\ &= \cos (p - \omega_m)t \cos \omega_m t - \sin (p - \omega_m)t \sin \omega_m t. \end{aligned}$$

In this way we obtain

$$V = A \cos \omega_m t - B \sin \omega_m t = R \cos (\omega_m t + \theta) \tag{3.10–22}$$

where A and B are relatively slowly varying functions of t given by

$$\begin{aligned} A &= P \cos (p - \omega_m)t + Q \cos (q - \omega_m)t \\ &\qquad + \sum_n c_n \cos (\omega_n t - \omega_m t - \varphi_n) \\ B &= P \sin (p - \omega_m)t + Q \sin (q - \omega_m)t \\ &\qquad + \sum_n c_n \sin (\omega_n t - \omega_m t - \varphi_n) \end{aligned} \tag{3.10–23}$$

and

$$\begin{aligned} R^2 &= A^2 + B^2, \qquad R > 0 \\ \tan \theta &= B/A \end{aligned} \tag{3.10–24}$$

As might be expected, (3.10–21) is closely associated with the problem of random flights and may be obtained from Kluyver's result[39] by assuming

[39] G. N. Watson, "Theory of Bessel Functions" (Cambridge, 1922), p. 420.

the noise to correspond to a very large number of very small random displacements.

Another way of deriving (3.10–21) is to assume $(p - \omega_m)t$, $(q - \omega_m)t$, φ_1, φ_2, $\cdots$ are independent random angles. The characteristic function of A, B is

$$\text{ave. } e^{iuA+ivB} = J_0(P\sqrt{u^2 + v^2})J_0(Q\sqrt{u^2 + v^2})e^{-(\psi_0/2)(u^2+v^2)}$$

The probability density of A, B is

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\, e^{-iuA-ivB} \text{ ave. } e^{iuA+ivB}$$

When the change of variables

$$A = R\cos\theta \qquad u = r\cos\varphi$$

$$B = R\sin\theta \qquad v = r\sin\varphi$$

is made the integration with respect to φ may be performed. The double integral becomes

$$\frac{1}{2\pi}\int_0^{\infty} rJ_0(Pr)J_0(Qr)J_0(Rr)e^{-(\psi_0/2)r^2}\,dr$$

This leads directly to (3.10–21) when we observe that $dA\,dB = RdRd\theta$.

Incidentally, if

$$I = Q(1 + k\cos pt)\cos qt + I_N$$

in which $p << q$, similar considerations show that the probability density of R is

$$\frac{R}{2\pi}\int_0^{2\pi} d\alpha \int_0^{\infty} rJ_0(Rr)J_0[Qr(1 + k\cos\alpha)]e^{-(\psi_0/2)r^2}\,dr$$

when ω_m is taken to be q. The integration with respect to r may be performed. This relation is closely connected with (3.10–11).

Returning now to the case in which I is the sum of two sine waves plus noise, we may show from (3.10–21) and

$$\int_0^{\infty} R^{n+1}J_0(Rr)\,dR = \frac{2^{n+1}\Gamma\left(1 + \frac{n}{2}\right)}{r^{n+2}\Gamma\left(-\frac{n}{2}\right)}$$

that the average value of R^n is, when $-2 < re\,(n) < -\frac{3}{2}$,

$$\overline{R^n} = \frac{2^{n+1}\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} \int_0^\infty r^{-n-1} J_0(Pr) J_0(Qr) e^{-\psi_0 r^2/2}\, dr$$

$$= (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right) \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_{k+m} (-x)^k (-y)^m}{k!\,k!\,m!\,m!} \qquad (3.10\text{–}25)$$

$$= (2\psi_0)^{n/2}\Gamma\left(\frac{n}{2}+1\right) \sum_{k=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_k (y-x)^k}{k!\,k!} P_k\left(\frac{x+y}{x-y}\right)$$

It appears very probable that this result could be extended, by analytic continuation, to positive integer values of n. We have used the notation

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$$
$$x = \frac{P^2}{2\psi_0}, \qquad y = \frac{Q^2}{2\psi_0} \qquad (3.10\text{–}26)$$

and have denoted the Legendre polynomial by $P_k(z)$. The series converge for all values of P, Q, and ψ_0 and terminate when n is an even positive integer.

When x or y, or both, are large in comparison with unity we may use the integral for $\overline{R^n}$ to obtain the asymptotic expansion, assuming $Q < P$ so that $y < x$,

$$\overline{R^n} \sim P^n \sum_{k=0}^{\infty} \frac{\left(-\frac{n}{2}\right)_k \left(-\frac{n}{2}\right)_k}{k!\,x^k} {}_2F_1\left(k-\frac{n}{2}, k-\frac{n}{2}; 1; \frac{y}{x}\right) \qquad (3.10\text{–}27)$$

When n is an even positive integer this series terminates and gives the same expression as (3.10–25). When n is an odd integer the ${}_2F_1$ may be expressed in terms of the complete elliptic functions E and K of modulus $y^{1/2}x^{-1/2}$:

$${}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right) = \frac{4}{\pi}E - \frac{2}{\pi}\left(1-\frac{y}{x}\right)K$$
$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{y}{x}\right) = \frac{2}{\pi}K \qquad (3.10\text{–}28)$$

The higher terms may be computed from

$$a(1-z)^2\,{}_2F_1(a+1, a+1; 1; z) = (2a-1)(1+z)\,{}_2F_1(a, a; 1; z)$$
$$+ (1-a)\,{}_2F_1(a-1, a-1; 1; z) \qquad (3.10\text{–}29)$$

which is a special case of

$$ab(\gamma+1)(1-z)^2\,{}_2F_1(a+1,b+1;c;z) = A\,{}_2F_1(a,b;c;z) - (\gamma-1)(c-a)(c-b)\,{}_2F_1(a-1,b-1;c;z) \quad (3.10\text{–}30)$$

where $\gamma = c - a - b$ and

$$A = (\gamma^2-1)\gamma + (1-z)[(\gamma-1)(c-b)(b-1) + (\gamma+1)a(c-a-1)]$$

Although this expression does not show it, A is really symmetrical in a and b. A symmetrical form may be obtained by using the expression obtained by putting $z = 0$ in (3.10–30).

3.11 Shot Effect Representation

In most of the work in this part the representations (2.8–1) or (2.8–6) have been used as a starting point. Here we point out that the shot effect representation used in Part I may also be used as a starting point.

For example, suppose we wish to find the two dimensional distribution of $I(t)$ and $I(t+\tau)$ discussed in Section 3.2. This is a special case of the distribution of the two variables

$$\begin{aligned} I(t) &= \sum_{k=-\infty}^{+\infty} F(t-t_k) \\ J(t) &= \sum_{k=-\infty}^{+\infty} G(t-t_k) \end{aligned} \quad (3.11\text{–}1)$$

where we now assume

$$\int_{-\infty}^{+\infty} F(t)\,dt = \int_{-\infty}^{+\infty} G(t)\,dt = 0 \quad (3.11\text{–}2)$$

in order that the average values of I and J may be zero. In fact, to get $I(t+\tau)$ from $J(t)$ we set $G(t)$ equal to $F(t+\tau)$.

The distribution of I and J may be obtained in much the same manner as was the distribution of I alone in section 1.4. The characteristic function of the distribution is

$$\begin{aligned} f(u,v) &= \text{ave. } e^{iuI+ivJ} \\ &= \exp\ \nu \int_{-\infty}^{+\infty} [e^{iuF(t)+ivG(t)} - 1]\,dt \end{aligned} \quad (3.11\text{–}3)$$

where ν is the expected number of events (electron arrivals in the shot effect) per second. The probability density of I and J is

$$\frac{1}{4\pi^2}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\, e^{-iuI-ivJ} f(u,v) \quad (3.11\text{–}4)$$

The semi-invariants $\lambda_{m,n}$ are given by the generating function

$$\log f(u, v) = \sum_{m,n=1}^{k} \frac{\lambda_{m,n}}{m!n!} (iu)^m (iv)^n + o[(iu)^k, (iv)^k]$$

and are

$$\lambda_{m,n} = \nu \int_{-\infty}^{+\infty} F^m(t) G^n(t)\, dt \tag{3.11–5}$$

As $\nu \to \infty$ the distribution of I and J approaches a two dimensional normal law. The approximation to this normal law may be obtained in much the same manner as in section 1.6. From our assumption (3.11–2) it follows that λ_{10} and λ_{01} are zero. From the relation between the second moments and semi-invariants λ we have

$$\begin{aligned} \mu_{11} &= \lambda_{20} + \lambda_{10}^2 = \nu \int_{-\infty}^{+\infty} F^2(t)\, dt \\ \mu_{12} &= \lambda_{11} + \lambda_{10}\lambda_{01} = \nu \int_{-\infty}^{+\infty} F(t)G(t)\, dt \\ \mu_{22} &= \lambda_{02} + \lambda_{01}^2 = \nu \int_{-\infty}^{+\infty} G^2(t)\, dt \end{aligned} \tag{3.11–6}$$

where the notation in the subscripts of the μ's differs from that of the λ's, the change being made to bring it in line with sections 2.9 and 2.10 so that we may write down the normal distribution at once.

The formulas (3.11–6) are closely related to Rowland's generalization of Campbell's theorem mentioned just below equation (1.5–9).

PART IV

NOISE THROUGH NON-LINEAR DEVICES

4.0 Introduction

We shall consider two problems which concern noise passing through detectors or other non-linear devices. The first deals with the statistical properties of the output of a non-linear device, that is, with its average value, its fluctuation about this average and so on. The second problem may be stated more definitely: Given a non-linear device and an input consisting of noise alone, or of noise plus a signal. What is the power spectrum of the output?

There does not seem to be much published material on the first problem. However, from conversation with other people, I have learned that it has been studied independently by several investigators. The same is probably true of the second problem although here the published material is somewhat more plentiful. This makes it difficult to assign credit where credit is due. Much of the material given here had its origin in discussions with friends, especially with W. R. Bennett, J. H. Van Vleck, and David Middleton. Help was obtained from the recent paper[37] by Bennett, and also from the manuscript of a forthcoming paper by Middleton.[40]

4.1 Low Frequency Output of a Square Law Device

Let the output current I of the device be related to the input voltage V by

$$I = \alpha V^2 \tag{4.1-1}$$

where α is a constant. When the power spectrum of V is confined to a relatively narrow band, the power spectrum of I consists of two portions. One portion clusters around twice the mid-band frequency of V and the other around zero frequency. We are interested in the low frequency portion. The current corresponding to this portion will be denoted by $I_{t\ell}$, and is the current which would flow if a low pass filter were inserted in the output to remove the upper portion of the spectrum. It is convenient to divide $I_{t\ell}$ into two components:

$$I_{t\ell} = I_{dc} + I_{\ell f} \tag{4.1-2}$$

[37] Loc. cit. (Section 3.10).

[40] Cruft Laboratory and the Research Laboratory of Physics, Harvard University, Cambridge, Mass. In the following sections references to Bennett's paper and Middleton's manuscript are made by simply giving the authors' names.

where the subscripts stand for "total low" frequency, "direct current." and "low frequency," respectively. We have

$$I_{dc} = \text{average } I_{t\ell} = \overline{I}_{t\ell} \tag{4.1-3}$$

$$\text{Mean Square } I_{\ell f} = \text{average } (I_{t\ell} - I_{dc})^2 = \overline{I^2_{t\ell}} - I^2_{dc}$$

Probably the simplest method of obtaining I_{dc} is to square the given expression for V and pick out the terms independent of time. Thus if

$$V = P \cos pt + Q \cos qt + V_N \tag{4.1-4}$$

we have

$$I_{dc} = \alpha\left(\frac{P^2}{2} + \frac{Q^2}{2} + \overline{V_N^2}\right) \tag{4.1-5}$$

$I_{\ell f}$ may also be obtained by picking out the low frequency terms. However, here we wish to use the square law device, and the linear rectifier in the next section, to illustrate a general method of dealing with the statistical properties of the output of a non-linear device when the input voltage is restricted to a relatively narrow band.

If none of the low frequency spectrum is removed by filters,

$$I_{t\ell} = \alpha \frac{R^2}{2} \tag{4.1-6}$$

where R is the envelope of V. The probability density and the statistical properties of $I_{t\ell}$ may be derived from this relation when the distribution function of R is known.[41] Before discussing these properties we shall establish (4.1–6).

Equation (4.1–6) is a special case of a more general result established in Section 4.3. However, its truth may be seen by taking the example

$$V = P \cos pt + Q \cos qt + V_N \tag{4.1-4}$$

where $f_p = p/2\pi$ and $f_q = q/2\pi$ lie within, or close to, the band of the noise voltage V_N.

By using formulas of the type

$$\begin{aligned} \cos pt &= \cos [(p - \omega_m)t + \omega_m t] \\ &= \cos (p - \omega_m)t \cos \omega_m t - \sin (p - \omega_m)t \sin \omega_m t \end{aligned} \tag{4.1-7}$$

[41] When part of the low-frequency spectrum is removed, the problem becomes much more difficult. I_{dc} may be obtained as above, but to get $\overline{I^2_{\ell f}}$ it is necessary to first determine the power spectrum of I (Section 4.5) and then integrate over the appropriate portion of it. Concerning the distribution of $I_{\ell f}$, our present knowledge tells us only that it lies between the one given by (4.1–6) and the normal law which it approaches when only a narrow portion of the low frequency spectrum is passed by the audio frequency filter (Section 4.3).

we may refer all terms to the mid-band frequency $f_m = \omega_m/2\pi$, as is done in equations (3.7–2) to (3.7–4).

In this way we obtain

$$V = A \cos \omega_m t - B \sin \omega_m t = R \cos (\omega_m t + \theta), \tag{4.1–8}$$

where A and B are relatively slowly varying functions of t given by

$$A = P \cos (p - \omega_m)t + Q \cos (q - \omega_m)t + \sum_n c_n \cos (\omega_n t - \omega_m t - \varphi_n),$$

$$B = P \sin (p - \omega_m)t + Q \sin (q - \omega_m)t + \sum_n c_n \sin (\omega_n t - \omega_m t - \varphi_n)$$

and

$$\begin{aligned} R^2 &= A^2 + B^2, \qquad R > 0 \\ \tan \theta &= B/A. \end{aligned} \tag{4.1–9}$$

This definition of R has also been given in equations (3.10–22, 23, 24).

The envelope of V is R and the output current is

$$I = \alpha R^2 \left[\frac{1}{2} + \frac{1}{2} \cos (2\omega_m t + 2\theta)\right] \tag{4.1–10}$$

Since R is a slowly varying function of time, so is R^2. The power spectrum of R^2 is confined to frequencies much lower than $2f_m$ and consequently the power spectrum of $R^2 \cos (2\omega_m t + 2\theta)$ is clustered around $2f_m$. Thus the only term in I contributing to the low frequency output is $\alpha R^2/2$ which is what we wished to show.

We now return to the statistical properties of $I_{t\ell}$. First, consider the case in which V consists of noise only, $V = V_N$, so that the probability density of the envelope R is

$$\frac{R}{\psi_0} e^{-R^2/2\psi_0} \tag{3.7–10}$$

where

$$\psi_0 = [rms\ V_N]^2 = \overline{V_N^2} \tag{4.1–11}$$

Hence

$$\begin{aligned} I_{dc} = \bar{I}_{t\ell} &= \frac{\alpha \overline{R^2}}{2} \\ &= \int_0^\infty \frac{\alpha R^2}{2} \frac{R}{\psi_0} e^{-R^2/2\psi_0}\, dR \\ &= \alpha\psi_0 \\ \overline{I_{tf}^2} = \overline{I_{t\ell}^2} - I_{dc}^2 &= \int_0^\infty \frac{\alpha^2 R^5}{4\psi_0} e^{-R^2/2\psi_0}\, dR - I_{dc}^2 \\ &= \alpha^2 \psi_0^2 \end{aligned} \tag{4.1–12}$$

Second, consider the case in which

$$V = V_N + P \cos pt \tag{4.1-13}$$

where $p/2\pi$ lies near the noise band of V_N. The probability density of the envelope R is

$$\frac{R}{\psi_0} \exp\left[-\frac{R^2 + p^2}{2\psi_0}\right] I_0\left(\frac{RP}{\psi_0}\right) \tag{3.10-11}$$

From this and equations (3.10–12), (3.10–13), we find

$$I_{dc} = \frac{\alpha \overline{R^2}}{2} = \alpha\psi_0 + \frac{\alpha P^2}{2} \tag{4.1-14}$$

$$\overline{I_{i\ell}^2} = \frac{\alpha^2}{4}\overline{R^4} = \alpha^2\left[2\psi_0^2 + 2P^2\psi_0 + \frac{P^4}{4}\right]$$

$$\overline{I_{\ell f}^2} = \overline{I_{i\ell}^2} - I_{dc}^2 = \alpha^2[\psi_0 + P^2]\psi_0 \tag{4.1-15}$$

In (4.1–14) ψ_0 is the mean square value of V_N and $P^2/2$ is the mean square value of the signal. These two equations show that I_{dc} and the rms value of $I_{\ell f}$ are independent of the distribution of the noise power spectrum in V_N as long as the input V is confined to a relatively narrow band. In other words, although this distribution does affect the power spectrum of the output, it does not affect the d.c. and rms $I_{\ell f}$ when ψ_0 and P are given. That the same is also true for a large class of non-linear devices was first pointed out by Middleton (see end of Section 4.9).

When the voltage is[42]

$$V = V_N + P \cos pt + Q \cos qt, \tag{4.1-4}$$

$p \neq q$, we obtain from equation (3.10–25)

$$I_{dc} = \frac{\alpha}{2}\overline{R^2} = \alpha\left(\psi_0 + \frac{P^2}{2} + \frac{Q^2}{2}\right)$$

$$\overline{I_{i\ell}^2} = \frac{\alpha^2}{4}\overline{R^4} \tag{4.1-16}$$

$$\overline{I_{\ell f}^2} = \alpha^2\left[\psi_0^2 + P^2\psi_0 + Q^2\psi_0 + \frac{P^2Q^2}{2}\right]$$

[42] These results are special cases, obtained by assuming no audio frequency filter, of formulas given by F. C. Williams, *Jour. Inst.* of *E. E.*, 80 (1937), 218–226. Williams also discusses the response of a linear rectifier to (4.1–4) when $P \gg Q + V_N$. An account of Williams' work is given by E. B. Moullin, "Spontaneous Fluctuations of Voltage," Oxford (1938), Chap. 7.

4.2 Low Frequency Output of a Linear Rectifier

In the case of the linear rectifier

$$I = \begin{cases} 0, & V < 0 \\ \alpha V, & V > 0 \end{cases} \tag{4.2-1}$$

the low frequency output current, assuming no audio frequency filter, is

$$I_{\ell\ell} = \frac{\alpha R}{\pi} \tag{4.2-2}$$

This formula, like its analogue (4.1–6) for the square law device, assumes that the applied signal and noise lie within a relatively narrow band. It may be used to compute the probability density and statistical properties of $I_{\ell\ell}$ when the corresponding information regarding the envelope R of the applied voltage is known.

The truth of (4.2–2) may be seen by considering the output I. It consists of the positive halves of the oscillations of αV. The envelope of I is the same as that of αV. However, the area under the loops of I is only about $1/\pi$ of the area under αR, this being the ratio of the area under a loop of $\sin x$ to the area of a rectangle of unit height and length 2π. From the low frequency point of view these loops of I merge into a current which varies as $\alpha R/\pi$.

When V is a sine wave plus noise,

$$V = V_N + P \cos pt \tag{4.1-13}$$

the average value of $I_{\ell\ell}$ is[43]

$$\begin{aligned} I_{dc} = \frac{\alpha}{\pi}\bar{R} &= \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2} {}_1F_1\left(-\frac{1}{2}; 1; -\frac{P^2}{2\psi_0}\right) \\ &= \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2} e^{-x/2}\left[(1+x)I_0\left(\frac{x}{2}\right) + xI_1\left(\frac{x}{2}\right)\right] \end{aligned} \tag{4.2-3}$$

where I_0, I_1 are Bessel functions of imaginary argument and

$$x = \frac{P^2}{2\psi_0} = \frac{\text{ave. sine wave power}}{\text{ave. noise power}} \tag{4.2-4}$$

[43] This result was discovered independently by several investigators, among whom we may mention W. R. Bennett and D. O. North. The latter has applied it to noise measurement work. He has found that the diode detector, when adapted to noise metering, is a great improvement over the thermocouple, and has used noise meters of this type satisfactorily since 1940. See D. O. North, "The Modification of Noise by Certain Non-Linear Devices", Paper read before I.R.E., Jan. 28, 1944.

ψ_0 being the average value of V_N^2. Equation (4.2–3) follows from the formulas (3.10–12) and (4B–9). When x is large the asymptotic expansion (4B–3) of the ${}_1F_1$ gives

$$I_{dc} \sim \frac{\alpha}{\pi}\left[P + \frac{\psi_0}{2P} + \frac{\psi_0^2}{8P^3} + \cdots\right] \tag{4.2–5}$$

Similarly, the mean square value of $I_{i\ell}$ is

$$\overline{I_{i\ell}^2} = \frac{\alpha^2}{\pi^2}\overline{R^2} = \frac{\alpha^2}{\pi^2}(P^2 + 2\psi_0) \tag{4.2–6}$$

and the mean square value of the low frequency current $I_{\ell f}$, excluding the d.c., is given by

$$\overline{I_{\ell f}^2} = \overline{I_{i\ell}^2} - I_{dc}^2$$

When x is large we have

$$\overline{I_{\ell f}^2} \sim \frac{\alpha^2}{\pi^2}\left[\psi_0 - \frac{\psi_0^2}{2P^2}\cdots\right] = \frac{\alpha^2}{\pi^2}\psi_0\left[1 - \frac{1}{4x}\cdots\right] \tag{4.2–7}$$

and when $x = 0$,

$$\overline{I_{\ell f}^2} = \frac{\alpha^2}{\pi^2}\psi_0\left(2 - \frac{\pi}{2}\right) \tag{4.2–8}$$

Curves for I_{dc} are given in Figures 1, 2 and 3 of Bennett's paper. He also gives curves, in Fig. 4, showing $\overline{I_{\ell f}^2}$ versus x. These show that the effect of the higher order modulation terms is small when $I_{\ell f}$ is computed by adding low frequency modulation products.

When V consists of two sine waves plus noise,

$$V = V_N + P\cos pt + Q\cos qt, \tag{4.1–4}$$

the average value of $I_{i\ell}$ is, from (3.10–25), a sort of double ${}_1F_1$ function:

$$\begin{aligned} I_{dc} &= \frac{\alpha}{\pi}\overline{R} = \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\tfrac{1}{2})_{k+m}}{k!\,k!\,m!\,m!}(-x)^k(-y)^m \\ &= \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2}\sum_{k=0}^{\infty}\frac{(-\tfrac{1}{2})_k}{k!\,k!}(y-x)^k P_k\left(\frac{x+y}{x-y}\right) \end{aligned} \tag{4.2–9}$$

where

$$x = \frac{P^2}{2\psi_0}, \qquad y = \frac{Q^2}{2\psi_0}, \qquad P_k(z) = \text{Legendre polynomial} \tag{4.2–10}$$

If x is large and $y < x$, we have from (3.10–27) the asymptotic expression

$$I_{dc} \sim \frac{\alpha}{\pi}P\sum_{k=0}^{\infty}\frac{(-\tfrac{1}{2})_k(-\tfrac{1}{2})_k}{k!\,x^k}\,{}_2F_1\left(k - \tfrac{1}{2}, k - \tfrac{1}{2}; 1; \frac{y}{x}\right) \tag{4.2–11}$$

The ${}_2F_1$ may be expressed in terms of the complete elliptic functions E and K of modulus $y^{1/2}x^{-1/2}$. Thus

$$
\begin{aligned}
{}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right) &= \frac{4}{\pi} E - \frac{2}{\pi}\left(1 - \frac{y}{x}\right) K, \\
{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{y}{x}\right) &= \frac{2}{\pi} K
\end{aligned}
\qquad (3.10\text{–}28)
$$

and the higher terms may be computed from the recurrence relation (3.10–29). The first term, $k = 0$, in (4.2–11) gives I_{dc} when the noise is absent.[44]

The mean square value of $I_{\ell\ell}$ is

$$
\overline{I_{\ell\ell}^2} = \frac{\alpha^2}{\pi^2}\overline{R^2} = \frac{\alpha^2}{\pi^2}[2\psi_0 + P^2 + Q^2] \qquad (4.2\text{–}14)
$$

From this expression and our expression for I_{dc}, the rms value of the low frequency current, $I_{\ell f}$, excluding the d.c., may be computed. For example, when the noise is small,

$$
\begin{aligned}
\overline{I_{\ell f}^2} \sim \frac{\alpha^2}{\pi^2}\Bigg[P^2 + Q^2 - \left(P\,{}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right)\right)^2 \\
+ 2\psi_0\left(1 - {}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{y}{x}\right)\frac{K}{\pi}\right)\Bigg]
\end{aligned}
\qquad (4.2\text{–}15)
$$

The term independent of ψ_0 gives the mean square low frequency current in the absence of noise. As Q goes to zero (4.2–15) approaches the leading term in (4.2–7), as it should. When $P = Q$ our formula breaks down and it appears that we need the asymptotic behavior of[45]

$$
I_{dc} = \alpha\left(\frac{\psi_0}{2\pi}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(-\tfrac{1}{2})_k (2k)!}{[k!]^4} (-x)^k
$$

In view of the questionable nature of the derivation given in Section 3.10 of equations (4.2–9) and (4.2–11) it was thought that a numerical check on their equivalence would be worth while. Accordingly, the values $x = 4$, $y = 3$ were used in the second series of (4.2–9). It was found that the largest term (about 130) in the summation occurred at $k = 11$. In all, 24 terms were taken. The result obtained was

$$
\frac{\overline{R}}{\sqrt{2\psi_0}} = 2.5502
$$

[44] See W. R. Bennett, *B.S.T.J.*, Vol. 12 (1933), 228–243.

[45] This may be done by the method given by W. B. Ford, Asymptotic Developments, Univ. of Mich. Press (1936), Chap. VI.

For the same values of x and y the asymptotic series (4.2–11) gave

$$2.40 + 0.171 + .075 + 0.52 + \cdots$$

If we stop just before the smallest term we get 2.57 for the sum. If we include the smallest term we get 2.65. This agreement indicates that (4.2–11) is actually the asymptotic expansion of (4.2–9).

When the voltage is of the form

$$V = Q(1 + k \cos pt) \cos qt + V_N$$

we may use

$$\overline{R^n} = (2\psi_0)^{n/2}\Gamma\left(1 + \frac{n}{2}\right)\frac{1}{2\pi}\int_0^{2\pi} {}_1F_1\left[-\frac{n}{2}; 1; -y(1 + k\cos\theta)^2\right] d\theta \tag{4.2–16}$$

where R is the envelope with respect to the frequency $q/2\pi$ and y is given by (4.2–10). The integral may be evaluated by writing ${}_1F_1$ as a power series and integrating termwise using the result

$$\frac{1}{2\pi}\int_0^{2\pi} (1 + k\cos\theta)^\ell \cos m\theta\, d\theta = \frac{(-\ell)_m}{2^m m!}(-k)^m {}_2F_1\left[\frac{m-\ell}{2}, \frac{m-\ell+1}{2}; m+1; k^2\right] \tag{4.2–17}$$

where m is a non-negative integer, ℓ any number,

$$(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1), \qquad (\alpha)_0 = 1, \qquad \text{and} \qquad (0)_0 = 1.$$

The integral may also be evaluated in terms of the associated Legendre function.

By applying the methods of Section 3.10 to (4.2–16) we are led to

$$\overline{R^2} = Q^2\left(1 + \frac{k^2}{2}\right) + 2\psi_0$$
$$\overline{R} \sim Q\sum_{s=0}^{\infty} \frac{(-\frac{1}{2})_s(-\frac{1}{2})_s}{s!\,y^s} {}_2F_1(s - \tfrac{1}{2}, s; 1; k^2) \tag{4.2–18}$$

where the asymptotic series holds when y is very large and k is not too close to unity. These expressions give

$$\overline{I_{lf}^2} \sim \frac{\alpha^2}{\pi^2}\left(Q^2\frac{k^2}{2} + \psi_0[2 - (1-k^2)^{-1/2}] + \cdots\right) \tag{4.2–19}$$

The reader might be tempted to associate the coefficient of ψ_0 in (4.2–19) with the continuous portion of the output power spectrum. However, this would not be correct. It appears that the principal contribution of the continuous portion of the power spectrum to $\overline{I_{\ell f}^2}$ is $\alpha^2\psi_0/\pi^2$, just as in (4.2–7) when k is zero. The difference between this and the corresponding term in (4.2–19) seems to arise from the fact that the amplitude of the recovered signal is not exactly $\alpha Qk/\pi$ but is modified by the presence of the noise. This general type of behavior might be expected on physical grounds since changing P, say doubling it, in (4.2–7) does not appreciably affect the $\overline{I_{\ell f}^2}$ in (4.2–7) (which is due entirely to the continuous portion of the noise spectrum). The modulating wave may be regarded as slowly making changes of this sort in P.

4.3 Some Statistical Properties of the Output of a General Non-Linear Device

Our general problem is this: Given a non-linear device whose output I is related to its input V by the relation

$$I = \frac{1}{2\pi}\int_C F(iu)e^{iVu}\,du \qquad (4A\text{–}1)$$

which is discussed in Appendix 4A. Let the input V contain noise in addition to the signal. Choose some frequency band in the output for study. What are the statistical properties of the current flowing in this band?

It seems to be difficult to handle this general problem. However, it appears that the two following results are true.

1. As the output band is chosen narrower and narrower the statistical properties of the corresponding current approach those of the random noise current discussed in Part III (provided no signal harmonic lies within the band). In particular, the instantaneous current values are distributed normally.

2. When the input V is confined to a relatively narrow band the power spectrum of the output I is clustered around the 0^{th} (d.c.), 1st, 2nd, etc. harmonics of the midband frequency of V. The low frequency output including the d.c. is

$$I_{\ell\ell} = A_0(R) = \frac{1}{2\pi}\int_C F(iu)J_0(uR)\,du \qquad (4.3\text{–}11)$$

where R is the envelope of V.

The envelope of the nth harmonic of the output, when $n > 0$, is

$$A_n(R) = \frac{1}{\pi}\int_C F(iu)J_n(uR)\,du \qquad (4.3\text{–}1)$$

The mathematical statement is

$$I = \sum_{n=0}^{\infty} A_n(R) \cos (n\omega_m t + n\theta) \tag{4.3-9}$$

where $f_m = \omega_m/(2\pi)$ is the representative mid-band frequency of V and θ is a relatively slowly varying phase angle. The results of Sections 4.1 and 4.2 are special cases of this.

Middleton's result that the noise power in each of the output bands (in the entire band corresponding to a given harmonic) depends only on $\overline{V_N^2} = \psi_0$ and not on the spectrum of V_N, where V_N is the noise voltage component of V, may also be obtained from (4.3-9). We note that the total power in the n^{th} band depends only on the mean square value of its envelope $A_n(R)$, and that the probability density of the envelope R of the input involves V_N only through ψ_0.

The argument we shall use in discussing the first result is not very satisfactory. It runs as follows. The output current I may be divided into two parts. One consists of sinusoidal terms due to the signal. The other consists of noise. We shall be concerned only with the latter which we shall call I_N. The correlation between two values of I_N separated by an interval of time approaches zero as the interval becomes large. Let τ be an interval long enough to ensure that the two values of I_N are substantially independent. Choose an interval of time T long enough to contain many intervals of length τ. Expand I_N as a Fourier series over this interval. We have

$$\begin{aligned} I_N &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right] \\ a_n - ib_n &= \frac{2}{T} \int_0^T e^{-i2\pi nt/T} I_N(t)\, dt \end{aligned} \tag{4.3-2}$$

Let the band chosen for study be $f_0 - \frac{\beta}{2}$ to $f_0 + \frac{\beta}{2}$ and let

$$T\left(f_0 - \frac{\beta}{2}\right) = n_1, \qquad T\left(f_0 + \frac{\beta}{2}\right) = n_2 \tag{4.3-3}$$

where n_1 and n_2 are integers. The number of components in the band is $(n_2 - n_1)$. We suppose β is such that this is small in comparison with T/τ. The output of the band is

$$J_N = \sum_{n=n_1}^{n_2} \left[a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi nt}{T} \right] \tag{4.3-4}$$

where

$$a_n - ib_n = \frac{2}{T}\int_0^T e^{-i2\pi((n/T)-f_0)t} e^{-i2\pi f_0 t} I_N(t)\, dt$$

$$n = \frac{n_1 + n_2}{2} + n - \frac{n_1 + n_2}{2} = f_0 T + (n - f_0 T) \tag{4.3-5}$$

We choose the band so narrow that

$$n_2 - n_1 \ll T/\tau \quad \text{or} \quad \beta\tau \ll 1 \tag{4.3-6}$$

This enables us to write approximately

$$a_n - ib_n = \sum_{r=1}^{r_1} e^{-i2\pi((n/T)-f_0)r\tau} \frac{2}{T}\int_{(r-1)\tau}^{r\tau} e^{-i2\pi f_0 t} I_N(t)\, dt$$

$r_1 = T/\tau$, T being chosen to make r_1 an integer. Suppose we do this for a large number of intervals of length T. Then $I_N(t)$ will differ from interval to interval. The set of integrals for $r = 1$ gives us an array of values which we regard as defining the distribution of a complex random variable, say x_1. Similarly the set of integrals for $r = 2$ defines the distribution of a second random variable x_2, and so on to x_{r_1}. Because we have chosen τ so large that $I_N(t)$ in any one integral is practically independent of its values in the other integrals we may say that $x_1, x_2, \cdots x_{r_1}$ are independent.

We have

$$a_{n_1} - ib_{n_1} = \sum_{r=1}^{r_1} e^{-i2\pi((n/T)-f_0)r\tau} x_r$$

$$a_{n_1+1} - ib_{n_1+1} = \sum_{r=1}^{r_1} e^{-i2\pi((n_1+1/T)-f_0)r\tau} x_r$$

$$\vdots \qquad\qquad \vdots$$

$$a_{n_2} - ib_{n_2} = \sum_{r=1}^{r_1} e^{-i2\pi((n_2/T)-f_0)r\tau} x_r$$

and if $n_2 - n_1 \ll r_1$, as was assumed in (4.3-6), we may apply the central limit theorem to show that $a_{n_1}, b_{n_1}, a_{n_1+1}, \cdots a_{n_2}, b_{n_2}$ tend to become independent and normally distributed about zero as we let the band width $\beta \to 0$ and $T \to \infty$ (and hence $r_1 \to \infty$) in such a way as to keep $n_2 - n_1$ fixed. In this work we make use of the fact that $I_N(t)$ is such that the real and imaginary parts of $x_1, x_2, \cdots x_r$ all have the same average and standard deviation. It is convenient to assume $f_0 T$ is an integer.

Thus as the band width β approaches zero the band output J_N given by (4.3-4) may be represented in the same way, namely as (2.8-1), as was the random noise current studied in Part III. Hence J_N tends to have the

same properties as the random noise current studied there. For example, the distribution of J_N tends towards a normal law. In our discussion we had to assume that $\beta\tau \ll 1$. If the voltage V applied to the non-linear device is confined to a relatively narrow frequency band, say $f_b - f_a$, it appears that the interval τ (chosen above so that $I(t)$ and $I(t + \tau)$ are substantially independent) may be taken to be of the order of $1/(f_b - f_a)$. In this case J_N tends to behave like a random noise current if $\beta/(f_b - f_a)$ is much smaller than unity.

We now turn our attention to the second statement made at the beginning of this section. Let the applied voltage be confined to a relatively narrow band so that it may be represented by equation (4.1-8) of Section 4.1,

$$V = R \cos (\omega_m t + \theta), \qquad R \geq 0, \tag{4.1-8}$$

where $f_m = \omega_m/(2\pi)$ is some representative frequency within the band and R and θ are functions of time which vary slowly in comparison with $\cos \omega_m t$. We call R the envelope of V.

From equation (4A-1)

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iuR \cos (\omega_m t + \theta)}\, du \tag{4.3-7}$$

We expand the integrand by means of

$$e^{ix \cos \varphi} = \sum_{n=0}^{\infty} \epsilon_n i^n \cos n\varphi J_n(x) \tag{4.3-8}$$

where ϵ_0 is 1 and ϵ_n is 2 when $n > 0$ and $J_n(x)$ is a Bessel function. Thus

$$I = \sum_{n=0}^{\infty} A_n(R) \cos (n\omega_m t + n\theta) \tag{4.3-9}$$

where

$$A_n(R) = \epsilon_n \frac{i^n}{2\pi} \int_C F(iu) J_n(uR)\, du \tag{4.3-10}$$

Since R is a relatively slowly varying function of time we expect the same to be true of $A_n(R)$, at least for moderately small values of n. Thus from (4.3-9) we see that the power spectrum of I will consist of a succession of bands, the n^{th} band being clustered around the frequency nf_m. If we eliminate all of the bands except the n^{th} by means of a filter we see that the output will have the envelope $A_n(R)$ when $n \geqq 1$. Taking n to be zero, shows that the low frequency output is simply

$$A_0(R) = \frac{1}{2\pi} \int_C F(iu) J_0(uR)\, du \tag{4.3-11}$$

Taking n to be one shows that the band around f_m is given by

$$\frac{A_1(R)}{R} V \tag{4.3-12}$$

The statistical properties of the low frequency output and of the envelopes of the output bands may be obtained from those of R. For example, the probability density of $A_n(R)$ is of the form

$$p(R) \Big/ \frac{dA_n(R)}{dR} \tag{4.3-13}$$

where $p(R)$ is the probability density of R. In this expression R is considered as a function of A_n.

It should be noted that we have been assuming that all of the band surrounding the harmonic frequency nf_m is taken. When we take only a portion of it, presumably the statistical properties will tend to approach those of a random noise current in accordance with the first statement made at the beginning of this section.

When we apply (4.3–11) to the square law device we have

$$F(iu) = \frac{2\alpha}{(iu)^3}$$

$$A_0(R) = -\frac{2\alpha}{2\pi i} \int^{(0+)} \frac{J_0(uR)}{u^3}\, du$$

$$= \frac{\alpha}{2} R^2$$

When we apply (4.3–11) to the linear rectifier:

$$F(iu) = -\frac{\alpha}{u^2}$$

$$A_0(R) = -\frac{\alpha}{2\pi} \int_{-\infty}^{+\infty} \frac{J_0(uR)}{u^2}\, du = \frac{\alpha R}{\pi}$$

where the path of integration passes under the origin. These two results agree with those obtained in Section 4.1 and 4.2 from simple considerations. As a final example we find the low frequency output of a biased linear rectifier in terms of the envelope R of the applied voltage. From the table of $F(iu)$ given in Appendix 4A we see that $F(iu)$ corresponding to

$$I = 0, \qquad V < B$$

$$I = V - B, \qquad V > B$$

is

$$F(iu) = -\frac{e^{-iuB}}{u^2}$$

Consequently, the low frequency output is

$$A_0(R) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iuB}J_0(uR)u^{-2}\,du$$

where the path of integration is indented downwards at the origin. When $B > R$ the value of the integral is zero since then the path of integration may be closed in the lower half plane by an infinite semi-circle This value also follows at once from the physics of the problem. When $-R < B < R$ we may integrate by parts and get

$$\begin{aligned} A_0(R) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iuB}[iBJ_0(uR) + RJ_1(uR)]u^{-1}\,du \\ &= -\frac{B}{2} + \frac{1}{\pi}\int_0^{\infty} [B\sin uBJ_0(uR) + R\cos uBJ_1(uR)]u^{-1}\,du \\ &= -\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2} \\ &= -\frac{B}{2} + \frac{R}{\pi}F\left(-\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}; \frac{B^2}{R^2}\right), \quad -R < B < R \end{aligned} \tag{4.3-14}$$

This hypergeometric function turns up again in equation (4.7-6). Also in the range $-R < B < R$,

$$\frac{dA_0}{dR} = \frac{1}{\pi}\sqrt{1 - \frac{B^2}{R^2}}$$

When B is negative and $R < -B$, the path of integration may be closed by an infinite semicircle in the upper half plane and the value of the integral is proportional to the residue of the pole at the origin:

$$\begin{aligned} A_0(R) &= 2\pi i\left(-\frac{1}{2\pi}\right)(-iB) \\ &= -B \end{aligned}$$

Thus, to summarize, the low frequency output for our linear rectifier is, for $B > 0$, (R is always positive)

$$\begin{aligned} A_0(R) &= 0, \qquad R < B \\ A_0(R) &= -\frac{B}{2} + \frac{B}{\pi}\arcsin\frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}, \qquad B < R \end{aligned} \tag{4.3-15}$$

and for $B < 0$ it is

$$A_0(R) = |B|, \quad R < |B|$$
$$A_0(R) = +\frac{|B|}{2} + \frac{|B|}{\pi} \arcsin \frac{|B|}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}, \quad |B| < R \tag{4.3–16}$$

where the arc sines lie between 0 and $\pi/2$. $A_0(R)$ and its first derivative with respect to R are continuous.

From (4.3–15), the d.c. output current is, for $B > 0$,

$$I_{dc} = \int_B^\infty \left[-\frac{B}{2} + \frac{B}{\pi} \arcsin \frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}\right] p(R)\, dR \tag{4.3–15}$$

where $p(R)$ is the probability density of the envelope of the input V, e.g., $p(R)$ is of the form (3.7–10) for noise alone, and of the form (3.10–11) for noise plus a sine wave. Similarly, the rms value of the low frequency current $I_{\ell f}$, excluding d.c., may be computed from

$$\overline{I_{\ell f}^2} = \overline{I_{i\ell}^2} - I_{dc}^2$$

where, if $B > 0$,

$$\overline{I_{i\ell}^2} = \int_B^\infty \left[-\frac{B}{2} + \frac{B}{\pi} \arcsin \frac{B}{R} + \frac{1}{\pi}\sqrt{R^2 - B^2}\right]^2 p(R)\, dR \tag{4.3–16}$$

If V consists of a sine wave of amplitude P plus noise V_N, so it may be represented as (4.1–13), and if $P \gg$ rms V_N, the distribution of R is approximately normal. If, in addition, $P - B \gg$ rms $V_N > 0$, (4.3–15), (4.3–16), and (3.10–19) lead to the approximations

$$I_{dc} \approx -\frac{B}{2} + \frac{B}{\pi} \arcsin \frac{B}{P} + \frac{1}{\pi}\sqrt{P^2 - B^2} + \frac{\psi_0}{2\pi\sqrt{P^2 - B^2}}$$
$$\approx -\frac{B}{2} + \frac{P}{\pi} + \frac{B^2 + \psi_0}{2\pi P} \tag{4.3–17}$$
$$\overline{I_{\ell f}^2} \approx \frac{P^2 - B^2}{\pi^2 P^2}\,\psi_0$$

The second expression for I_{dc} assumes $P \gg B$. When $B = 0$, these reduce to the first terms of (4.2–5) and (4.2–7). By using a different method Middleton has obtained a more precise form of this result.

Incidentally, for a given applied voltage, $I_{dc}(+)$ for a positive bias $|B|$ is related to $I_{dc}(-)$ for a negative bias $-|B|$ by

$$I_{dc}(-) = |B| + I_{dc}(+) \tag{4.3–18}$$

Also r.m.s. $I_{\ell f}(+)$ is equal to r.m.s. $I_{\ell f}(-)$. Equation (4.3–18) follows from a physical argument based on the areas underneath a curve of I for

the two cases. Both of the above relations follow from formulas given by Middleton when V is the sum of a sine wave plus noise. They may also be derived from (4.3–15) and (4.3–16).

4.4 Output Power Spectrum

The remainder of Part IV will be concerned with methods of solving the following problem: Given a non-linear device and an input voltage consisting of noise alone or of a signal plus noise. What is the power spectrum of the output?

In some ways the answer to this problem gives us less information than the methods discussed in the first three sections. For example, beyond giving the rms value, it tells us very little about the probability density of the current corresponding to a given frequency band of the output. On the other hand, this rms value may be found (by integrating the power spectrum) for any band we choose to study. The methods described earlier depended on the input being confined to a relatively narrow band and gave information regarding only the entire band corresponding to a given harmonic (0th, 1st, 2nd, etc.) of the input. There was no way to study the output when part of a band was eliminated by filters except by obtaining the power spectrum of some function of the envelope.

At present there appear to be two general methods available for the determination of the output power spectrum each with its own advantages and disadvantages. First there is the direct method which has been used by W. R. Bennett*, F. C. Williams**, J. R. Ragazzini[46] and others. The noise is represented as the sum of a finite number of sinusoidal components. The typical modulation product is computed and the output power spectrum is obtained by considering the density and amplitude of these products. The chief advantage of this method lies in its close relation to the known theory of modulation in non-linear circuits. Generally, the lower order modulation products are the only ones which contribute significantly to the output power and when they are known, the problem is well along towards solution. The main disadvantage is the labor of counting the modulation products falling in a given interval. However, Bennett has developed a method for doing this.[47]

The fundamental idea of the second method is to obtain the correlation function for the output current. From this the output power spectrum may be obtained by Fourier's transform. The correlation function method and its variations are of more recent origin than the direct method. They have

* Cited in Section 4.0. Also much of this writer's work on interference in broad band communication systems may be carried over to noise theory without any change in the methods used.

** Cited in Section 4.1.

[46] *Proc. I.R.E.* Vol. 30, pp. 277–288 (June 1942), "The Effect of Fluctuation Voltages on the Linear Detector."

[47] *B.S.T.J.*, Vol. 19 (1940), pp. 587–610, Appendix B.

been discovered independently and at about the same time, by several workers. In a paper read before the I.R.E., Jan. 28, 1944, D. O. North described results obtained by using the correlation function. J. H. Van Vleck and D. Middleton have been using the two variations of the method which we shall describe in Sections 4.7 and 4.8, since early in 1943. A primitive form of the method of Section 4.8 had been used by A. D. Fowler and the writer in some unpublished material written in 1942. Recently, I have learned that a method similar to the one used by Fowler and myself had already been used by Kurt Fränz in 1941.[48]

The correlation function method avoids the problem of counting the modulation products. However, in some cases it becomes rather unwieldy. Probably it is best to have both methods in mind when investigating any particular problem. The direct method will be illustrated by applying it to the square law detector. Two approaches to the correlation function method will then be described and applied to examples.

4.5 Noise Through Square Law-Device

Probably the most direct method of obtaining the power spectrum $W(f)$ of I, where

$$I = \alpha V^2, \tag{4.1-1}$$

V being a noise voltage, is to square the expression

$$V = V_N = \sum_1^M c_m \cos(\omega_m t - \varphi_m) \tag{2.8-6}$$

in which c_m^2 is $2w(f_m)\Delta f$, $\omega_m = 2\pi f_m$, $f_m = m\Delta f$ and $\varphi_1, \varphi_2, \cdots \varphi_M$ are random phase angles.

Considerable simplification of the algebra results when we replace the representation (2.8–6) by

$$V_N = \frac{1}{2}\sum_{-\infty}^{\infty} c_m e^{ima t - i\varphi_m} \tag{4.5-1}$$

Here we have added a term $c_0/2$ so as to not have any gaps in the summation and have introduced the definitions

$$\begin{aligned} c_{-m} &= c_m \\ \varphi_{-m} &= -\varphi_m \\ a &= 2\pi\Delta f \end{aligned} \tag{4.5-2}$$

[48] "Die Übertragung von Rauschspannung über den linearen Gleichrichter," *Hochfr. u. Elektroakust.*, June 1941. Other articles by Fränz are (I am indebted to Dr. North for the following references) "Beitrage zur Berechnung des Verhaltnisses von Signal spannung zu Rauschspannung am Ausgang von Empfängern", *E.N.T.*, 17, 215, 1940 and 19, 285, 1942. "Die Amplituden von Geräuschspannungen", *E.N.T.*, 19, 166, 1942. The May 1944 (p. 237), issue of the Wireless Engineer contains an abstract of "The Influence of Carrier Waves on the Noise on the Far Side of Amplitude-Limiters and Linear Rectifiers" by Fränz and Vellat, *E.N.T.*, Vol. 20, pp. 183–189 (Aug. 1943).

Squaring (4.5–1) gives the double series

$$V_N^2 = \frac{1}{4}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} c_m c_n e^{i(m+n)at - i\varphi_m - i\varphi_n}$$

$$= \frac{1}{4}\sum_{k=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} c_{k-n} c_n e^{ikat - i\varphi_{k-n} - i\varphi_n}$$

Suppose we wish to consider the component of V_N^2 of frequency $f_k = k\Delta f$. It is seen to be

$$A_k \cos(\omega_k t - \psi_k) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} c_{k-n} c_n \cos(kat - \varphi_{k-n} - \varphi_n) \qquad (4.5\text{–}3)$$

The power spectrum $W(f)$ of I at frequency f_k is α^2 times the coefficient of Δf in the mean square value of (4.5–3) where the average is taken over the φ's. Thus

$$W(f_k)\Delta f = \frac{\alpha^2}{4}\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} c_{k-n} c_n c_{k-m} c_m$$

$$\times \text{ ave. } \cos(kat - \varphi_{k-n} - \varphi_n)\cos(kat - \varphi_{k-m} - \varphi_m)$$

where the summations extend over m and n. Let n be fixed and consider those values of m which give an average different from zero. We see that $m = n$ and $m = k - n$ are two such values. The only other possibilities are $m = -n$ and $m = -k + n$, but these lead to terms containing (except when n or k equal zero) three different angles, φ_n, φ_{k-n}, and φ_{k+n} which average to zero. Using the fact that the average of cosine squared is one-half and that for a given n there are two such terms, we get

$$W(f_k)\Delta f = \frac{\alpha^2}{4}\sum_{n=-\infty}^{+\infty} c_{k-n}^2 c_n^2$$

$$= \alpha^2 \Delta f \sum_{n=-\infty}^{+\infty} w(f_k - f_n) w(f_n)\Delta f \qquad (4.4\text{–}5)$$

where in the last step we have used

$$f_{k-n} = (k - n)\Delta f = f_k - f_n$$

and have implied, from $c_{-n} = c_n$, that

$$w(f_{-n}) = w(-n\Delta f) = w(-f_n)$$

is equal to $w(f_n)$.

Thus, from (4.5–4), we get for the power spectrum of I

$$W(f) = \alpha^2 \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \qquad (4.5\text{–}5)$$

with the understanding that f is not zero and

$$w(-x) = w(x). \tag{4.5-6}$$

The result which is obtained by using (2.8–6), involving the cosines and only positive values of m, is

$$W(f) = \alpha^2 \int_0^f w(x)w(f - x)\,dx + 2\alpha^2 \int_0^\infty w(x)w(f + x)\,dx \tag{4.5-7}$$

This contains only positive values of frequency. (4.5–5) and (4.5–7) are equivalent and may readily be transformed into each other.

The first integral in (4.5–7) arises from second order modulation products of the sum type and the second integral from products of the difference type. This may be seen by writing the current as

$$\begin{aligned} I = \alpha V^2 &= \alpha \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \cos(\omega_m t - \varphi_m) \cos(\omega_n t - \varphi_n) \\ &= \frac{\alpha}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_m c_n \{\cos[(\omega_m - \omega_n)t - \varphi_m + \varphi_n] \\ &\qquad + \cos[(\omega_m + \omega_n)t + \varphi_m + \varphi_n]\} \end{aligned} \tag{4.5-8}$$

The power in the range $f_k, f_k + \Delta f$ is the power due to modulation products of the difference type, $\omega_{k+\ell} - \omega_\ell$, plus the power due to the modulation products of the sum type, $\omega_{k-\ell} + \omega_\ell$. In the first type ℓ runs from 1 to ∞ and in the second type ℓ runs from 1 to $k - 1$.

Consider the difference type first, and for the moment take both k and ℓ to be fixed. The two sets $m = k + \ell$, $n = \ell$ and $m = \ell$, $n = k + \ell$ are the only values of m and n in (4.5–8) leading to $\omega_{k+\ell} - \omega_\ell$. The two corresponding terms in (4.5–8) are equal because $\cos(-x)$ is equal to $\cos x$. The average power contributed by these two terms is

$$\begin{aligned} \left(\frac{\alpha}{2} c_{k+\ell}\, c_\ell\right)^2 \times \{\text{Average of } (2\cos[(\omega_{k+\ell} - \omega_\ell)t - \varphi_{k+\ell} + \varphi_\ell])^2\} \\ = \tfrac{1}{2}(\alpha c_{k+\ell}\, c_\ell)^2 \end{aligned} \tag{4.5-9}$$

The power contributed to $f_k, f_k + \Delta f$ by the difference modulation products is obtained by summing ℓ from 1 to ∞:

$$\begin{aligned} \frac{\alpha^2}{2} \sum_{\ell=1}^{\infty} c_{k+\ell}^2 c_\ell^2 &= 2\alpha^2 \sum_{\ell=1}^{\infty} w(f_{k+\ell})w(f_\ell)(\Delta f)^2 \\ &\to 2\alpha^2 \Delta f \int_0^\infty w(f_k + f)w(f)\,df \end{aligned}$$

This leads to the second term in (4.5–7).

Now consider the modulation products of the sum type. The terms of this type in (4.5–8) which give rise to the frequency ω_k are those for which $m + n$ is equal to k. Let n be 1 then $m = k - 1$. The phase of this term is random with respect to all the other terms except the one given by $n = k - 1$, $m = 1$ which has the same phase. The average power contributed by these two terms in (4.5–8) is, as in (4.5–9),

$$\tfrac{1}{2}(\alpha c_1 c_{k-1})^2$$

This disposes of two terms for which $m + n$ is equal to k. Taking n to be 2 and going through the same process gives two more. Thus, assuming for the moment that k is an odd number, the power contributed to the interval $f_k, f_k + \Delta f$ by the sum modulation products is

$$\frac{1}{2}\sum_{n=1}^{(k-1)/2}(\alpha c_n c_{k-n})^2 = \frac{1}{4}\sum_{n=1}^{k-1}(\alpha c_n c_{k-n})^2 \to \alpha^2 \Delta f \int_0^{f_k} w(f) w(f_k - f)\, df$$

and this leads to the second term in (4.5–7).

When the voltage V applied to the square law device is the sum of a noise voltage V_N and a sine wave:

$$V = P \cos pt + V_N, \tag{4.1–13}$$

we have

$$V^2 = P^2 \cos^2 pt + 2PV_N \cos pt + V_N^2 \tag{4.5–10}$$

From the two equations

$$\cos^2 pt = \frac{1}{2} + \frac{1}{2}\cos 2pt$$

$$\text{ave. } V_N^2 = \sum_1^M c_m^2 \frac{1}{2} \to \int_0^\infty w(f)\, df$$

we see that I, or αV^2, has a dc component of

$$\frac{\alpha P^2}{2} + \alpha \int_0^\infty w(f)\, df \tag{4.5–11}$$

which agrees with (4.1–14), and a sinusoidal component

$$\frac{\alpha P^2}{2} \cos 2pt \tag{4.5–12}$$

The continuous power spectrum $W_c(f)$ of the remaining portion of I may be computed from

$$2PV_N \cos pt + V_N^2.$$

Using the representation (2.8–6) we see

$$2PV_N \cos pt = P \sum_1^M c_m[\cos(\omega_m t + pt - \varphi_m) + \cos(\omega_m t - pt - \varphi_m)]$$

For the moment, we take $p = 2\pi r\Delta f$. The terms pertaining to frequency $f_n = n\Delta f$ are those for which

$$\begin{aligned} \omega_m + p &= 2\pi f_n & |\omega_m - p| &= 2\pi f_n \\ m + r &= n & |m - r| &= n \\ m &= n - r & m &= r \pm n \end{aligned}$$

where only positive values of m are to be taken: If $n > r$, then m is $n - r$ or $r + n$. If $n < r$, then m is $r - n$ or $r + n$. In either case the values of m are $|n - r|$ and $n + r$. The terms of frequency f_n in $2PV_N \cos pt$ are therefore

$$Pc_{|n-r|} \cos(2\pi f_n t - \varphi_{|n-r|}) + Pc_{n+r} \cos(2\pi f_n t - \varphi_{n+r})$$

and the mean square value of this expression, the average being taken over the φ's, is

$$\begin{aligned} \frac{P^2}{2}(c^2_{|n-r|} + c^2_{n+r}) &= P^2\Delta f[w(f_{|n-r|}) + w(f_{n+r})] \\ &= P^2\Delta f[w(|f_n - f_p|) + w(f_n + f_p)] \end{aligned}$$

where f_p denotes $p/2\pi$.

By combining this with the expression (4.5–5) which arises from V_N^2 we see that the continuous portion $W_c(f)$ of the power spectrum of I is

$$\begin{aligned} W_c(f) = \alpha^2 P^2[w(f - f_p) + w(f + f_p)] \\ + \alpha^2 \int_{-\infty}^{+\infty} w(x)w(f - x)\,dx \end{aligned} \qquad (4.5\text{–}13)$$

where $w(-f)$ has the same value as $w(f)$.

Equation (4.5–13) has been used to compute $W_c(f)$ as shown in Fig. 8. The input noise is assumed to be uniform over a band of width β centered at f_p, cf. Filter c, Appendix C. By noting the area under the low frequency portion of the spectrum we find

$$\int_0^\beta W_c(f)\,df = \alpha^2\beta w_0(P^2 + \beta w_0)$$

Since the mean square value of the input V_N is $\psi_0 = \beta w_0$, it is seen that this equation agrees with the expression (4.1–15) for the mean square value of $I_{\ell f}$, the low frequency current, excluding the d.c. If audio frequency

filters cut out part of the spectrum, $W_c(f)$ may be integrated over the remaining portion to give the mean square value of the corresponding output current. This idea is mentioned in the footnote pertaining to equation (4.1–6).

If V consists of V_N plus two sinusoidal voltages of incommensurable frequencies, say

$$V = P \cos pt + Q \cos qt + V_N ,$$

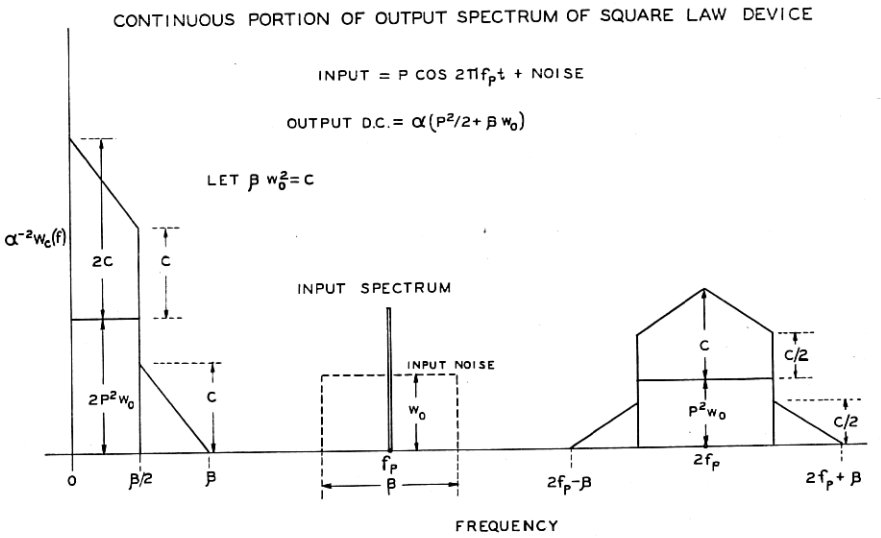

Fig. 8

the continuous portion $W_c(f)$ of the power spectrum of I may be shown to be (4.5–13) plus the additional terms

$$\alpha^2 Q^2 [w(f - f_q) + w(f + f_q)] \tag{4.5–14}$$

where f_q denotes $q/2\pi$.

When the voltage applied to the square law device (4.1–1) is[49]

$$V(t) = Q(1 + k \cos pt) \cos qt + V_N$$

$$= Q \cos qt + \frac{Qk}{2} \cos (p + q)t + \frac{Qk}{2} \cos (p - q)t + V_N$$

the resulting current contains the dc component

$$\frac{\alpha}{2} Q^2 \left(1 + \frac{k^2}{2}\right) + \alpha \int_0^\infty w(f)\, df \tag{4.5–16}$$

[49] A complete discussion of this problem is given by L. A. MacColl in a manuscript being prepared for publication.

The sinusoidal terms of I are obtained by squaring

$$Q(1 + k \cos pt) \cos qt$$

and multiplying by α. The remaining portion of I has a continuous power spectrum given by

$$\begin{aligned} W_c(f) = \alpha^2 Q^2 \Big[& w(f - f_q) + w(f + f_q) \\ & + \frac{k^2}{4} w(f - f_p - f_q) + \frac{k^2}{4} w(f + f_p + f_q) \\ & + \frac{k^2}{4} w(f - f_p + f_q) + \frac{k^2}{4} w(f + f_p - f_q) \Big] \\ & + \alpha^2 \int_{-\infty}^{+\infty} w(x) w(f - x)\, dx \end{aligned} \tag{4.5–17}$$

where f_p denotes $p/2\pi$ and f_q denotes $q/2\pi$.

4.6 Two Correlation Function Methods

As mentioned in Section 4.4 these methods for determining the output power spectrum are based on finding the correlation function $\Psi(\tau)$ for the output current. From this the power spectrum, $W(f)$, of the output current may be obtained from (2.1–5), rewritten as

$$W(f) = 4 \int_0^{\infty} \Psi(\tau) \cos 2\pi f\tau \, d\tau \tag{4.6–1}$$

It will be recalled that $W(f)\Delta f$ may be regarded as the average power which would be dissipated by those components of I in the band $f, f + \Delta f$ if I were to flow through a resistance of one ohm.

The input of the non-linear device is taken to be a voltage $V(t)$. It may, for example, consist of a noise voltage $V_N(t)$ plus sinusoidal components. The output is taken to be a current $I(t)$. The non-linear device is specified by a relation between $V(t)$ and $I(t)$. In this work $I(t)$ at time t is assumed to be completely determined by the value of $V(t)$ at time t.

Two methods of obtaining $\Psi(\tau)$ will be described.

(a) Integrating the two-dimensional probability density of $V(t)$ and $V(t + \tau)$ over the values allowed by the non-linear device. This method, which is especially direct when applied to noise alone through rectifiers, was discovered independently by Van Vleck and North.

(b) Introducing and using the characteristic function, which for the sake of brevity will be abbreviated to ch. f., of the two-dimensional probability distribution of $V(t)$ and $V(t + \tau)$.

4.7 Linear Detection of Noise—The Van Vleck-North Method

The method due to Van Vleck and North will be illustrated by using it to determine the output power spectrum of a linear detector when the input consists of noise alone.

The linear detector is specified by

$$I(t) = \begin{cases} 0, & V(t) < 0 \\ V(t), & V(t) > 0, \end{cases} \tag{4.7-1}$$

which may be obtained from (4.2–1) by setting α equal to one, and the input voltage is

$$V(t) = V_N(t) \tag{4.7-2}$$

where $V_N(t)$ is a noise voltage whose correlation function is $\psi(\tau)$ and whose power spectrum is $w(f)$.

The correlation function $\Psi(\tau)$ is the average value of $I(t)I(t+\tau)$. This is the same as the average value of the function

$$F(V_1, V_2) = \begin{cases} V_1 V_2, & \text{when both } V_1, V_2 > 0 \\ 0, & \text{all other } V\text{'s}, \end{cases} \tag{4.7-3}$$

where we have set

$$V_1 = V(t)$$

$$V_2 = V(t+\tau)$$

The two-dimensional distribution of V_1 and V_2 is given by (3.2–4), and from this it follows that the average value of any function $F(V_1, V_2)$ is

$$\int_{-\infty}^{+\infty} dV_1 \int_{-\infty}^{+\infty} dV_2 \frac{F(V_1, V_2)}{2\pi |M|^{1/2}} \exp\left[-\frac{1}{2|M|}(\psi_0 V_1^2 + \psi_0 V_2^2 - 2\psi_\tau V_1 V_2)\right] \tag{4.7-4}$$

where

$$|M| = \psi_0^2 - \psi_\tau^2 .$$

For the linear rectifier case, where $F(V_1, V_2)$ is given by (4.7-3), the integral is

$$|M|^{-1/2} \frac{1}{2\pi} \int_0^\infty dV_1 \int_0^\infty dV_2 V_1 V_2 \exp\left[-\frac{1}{2|M|}(\psi_0 V_1^2 + \psi_0 V_2^2 - 2\psi_\tau V_1 V_2)\right]$$

$$= \frac{1}{2\pi}\left([\psi_0^2 - \psi_\tau^2]^{1/2} + \psi_\tau \cos^{-1}\left[\frac{-\psi_\tau}{\psi_0}\right]\right)$$

where we have used (3.5–4) to evaluate the integral. The arc cosine is taken to be between 0 and π. We therefore have for the correlation function of $I(t)$,

$$\Psi(\tau) = \frac{1}{2\pi}\left([\psi_0^2 - \psi_\tau^2]^{1/2} + \psi_\tau \cos^{-1}\left[\frac{-\psi_\tau}{\psi_0}\right]\right) \tag{4.7–5}$$

The power spectrum $W(f)$ may be obtained from this by use of (4.6–1). For this purpose it is convenient to write (4.7–5) in terms of a hypergeometric function. By expanding and comparing terms it is seen that

$$\begin{aligned}\Psi(\tau) &= \frac{\psi_\tau}{4} + \frac{\psi_0}{2\pi} F\left(-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \frac{\psi_\tau^2}{\psi_0^2}\right) \\ &= \frac{\psi_\tau}{4} + \frac{\psi_0}{2\pi} + \frac{\psi_\tau^2}{4\pi\psi_0} + \text{terms involving } \psi_\tau^4, \psi_\tau^6, \text{ etc.}\end{aligned} \tag{4.7–6}$$

As will be discussed more fully in Section 4.8, a constant term A^2 in $\psi(\tau)$ indicates a direct current component of $I(t)$ of A amperes. Thus $I(t)$ has a *dc* component equal to

$$\left[\frac{\psi_0}{2\pi}\right]^{1/2} = \frac{1}{\sqrt{2\pi}} \times \text{rms value of } V(t) \tag{4.7–7}$$

This agrees with (4.2–3) when the P of that equation is set equal to zero.

Integrals of the form

$$G_n(f) = \int_0^\infty \psi_\tau^n \cos 2\pi f\tau \, d\tau$$

which result when (4.7–6) is put in (4.6–1) and integrated termwise are discussed in Appendix 4C. From the results given there it is seen that if we neglect ψ_τ^4 and higher powers we obtain an approximation for the continuous portion $W_c(f)$ of $W(f)$:

$$\begin{aligned}W_c(f) &\approx G_1(f) + \frac{G_2(f)}{\pi\psi_0} \\ &= \frac{w(f)}{4} + \frac{1}{4\pi\psi_0}\cdot\frac{1}{2}\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx\end{aligned} \tag{4.7–8}$$

where $w(-f)$ is defined as $w(f)$.

When $V_N(t)$ is uniform over a relatively narrow band extending from f_a to f_b so that $w(f)$ is equal to w_0 in this band and is zero outside it, we may use the results for Filter c of Appendix 4C. The f_0 and β given there are related to f_a and f_b by

$$f_a = f_0 - \frac{\beta}{2}, \qquad f_b = f_0 + \frac{\beta}{2}$$

and the value of w_0 taken there is the same as here and is ψ_0/β. The value of $G_2(f)$ given there leads to the approximation, for low frequencies:

$$W_c(f) \approx \frac{1}{\pi\psi_0}\frac{\psi_0^2}{4\beta}\left(1 - \frac{f}{\beta}\right) = \frac{w_0}{4\pi}\left(1 - \frac{f}{f_b - f_a}\right) \tag{4.7–9}$$

when $0 < f < f_b - f_a$, and to $W_0(f) \approx 0$ for $f_b - f_a < f < f_a$. By setting P equal to zero in the curve given in Fig. 8 for $W_c(f)$ corresponding to the square law detector, we see that the low frequency portion of the power spectrum is triangular in shape and is zero at $f = \beta$. Thus, looking at (4.7–9), we see that to a first approximation the shape of the output power spectrum is the same for a linear detector as for a square law detector when the input consists of a relatively narrow band of noise.

An approximate rms value of the low frequency output current may be obtained by integrating (4.7–9)

$$\overline{I_{lf}^2} = \int_0^{f_b - f_a} W_c(f)\, df \approx \frac{w_0(f_b - f_a)}{8\pi} = \frac{\psi_0}{8\pi}$$

$$\text{rms low freq. current} \approx \frac{1}{\sqrt{8\pi}} \times \text{rms applied voltage} \tag{4.7–10}$$

It is seen that this is half of the direct current. It must be kept in mind that (4.7–10) is an approximation because we have neglected ψ_τ^4 and higher powers. The true value may be obtained from (4.2–8). It is seen that the coefficient $(8\pi)^{-1/2} = 0.200$ should be replaced by

$$\frac{1}{\pi}\left(2 - \frac{\pi}{2}\right)^{1/2} = 0.209$$

$W_c(f)$ for other types of band pass filters may be obtained by using the corresponding G's given in appendix 4C. It turns out that (4.7–10) holds for all three types of filters. This is a special case of Middleton's theorem, mentioned several times before, that the total power in any modulation product (it will be shown later in Section 4.9 that the term ψ_τ^n in (4.7–6) corresponds to the n^{th} order modulation products) depends only on the total input power of the applied noise, not on its spectral distribution.

4.8 The Characteristic Function Method

As mentioned in the preceding parts, especially in connection with equation (1.4–3), the ch. f. of a random variable x is the average value of exp

(iux). This is a function of u. The ch. f. of two random variables x and y is the average value of exp $(iux + ivy)$ and is a function of u and v. The ch. f. which we shall use here is the ch. f. of the two random variables $V(t)$ and $V(t + \tau)$ where $V(t)$ is the voltage applied to the non-linear device, and the randomness is introduced by t being selected at random, τ remaining fixed. We may write this characteristic function as

$$g(u, v, \tau) = \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T \exp\left[iuV(t) + ivV(t + \tau)\right] dt \quad (4.8\text{–}1)$$

If $V(t)$ contains a noise voltage $V_N(t)$, as it always does in this section, and if we use the representation (2.8–1) or (2.8–6) a large number of random parameters (a_n's and b_n's or φ_n's) will appear in (4.8–1). In accordance with our use of such representations we may average over these parameters without changing the value of (4.8–1) and may thereby simplify the integration.

For example suppose

$$V(t) = V_s(t) + V_N(t) \quad (4.8\text{–}2)$$

where $V_s(t)$ is some regular voltage which may, e.g., consist of one or more sine waves. Substituting this in (4.8–1) and using the result (3.2–7) that the ch. f. of $V_N(t)$ and $V_N(t + \tau)$ is

$$\begin{aligned} g_N(u, v, \tau) &= \text{ave. exp}\left[iuV_N(t) + ivV_N(t + \tau)\right] \\ &= \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \end{aligned} \quad (4.8\text{–}3)$$

$\psi_\tau \equiv \psi(\tau)$ being the correlation function of $V_N(t)$, we obtain for the ch. f. of $V(t)$ and $V(t + \tau)$,

$$\begin{aligned} g(u, v, \tau) &= \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \\ &\quad \times \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T \exp\left[iuV_s(t) + ivV_s(t + \tau)\right] dt \\ &= g_N(u, v, \tau) g_s(u, v, \tau) \end{aligned} \quad (4.8\text{–}4)$$

In the last line we have used $g_s(u, v, \tau)$ to denote the limit in the line above:

$$g_s(u, v, \tau) = \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T \exp\left[iuV_s(t) + ivV_s(t + \tau)\right] dt \quad (4.8\text{–}5)$$

The principal reason we use the ch. f. is because quite a few non-linear devices may be described by the integral

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu}\, du \quad (4A\text{–}1)$$

where the function $F(iu)$ and the path of integration C are chosen to fit the device. Examples of such devices are given in Appendix 4A. The correlation function $\Psi(\tau)$ of $I(t)$ is given by

$$\begin{aligned}\Psi(\tau) &= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt \\ &= \operatorname*{Limit}_{T\to\infty} \frac{1}{4\pi^2 T}\int_0^T dt \int_C F(iu)e^{iuV(t)}\,du \int_C F(iv)e^{ivV(t+\tau)}\,dv \\ &= \frac{1}{4\pi^2}\int_C F(iu)\,du \int_C F(iv)\,dv \qquad (4.8\text{–}6) \\ &\qquad \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T \exp\left[iuV(t) + ivV(t+\tau)\right] dt \\ &= \frac{1}{4\pi^2}\int_C F(iu)\,du \int_C F(iv)g(u, v, \tau)\,dv\end{aligned}$$

This is the fundamental formula of the ch. f. method.

When $V(t)$ is the sum of a noise voltage and a regular voltage, as in (4.8–2), (4.8–6) becomes

$$\begin{aligned}\Psi(\tau) = \frac{1}{4\pi^2}\int_C F(iu)e^{-(\psi_0/2)u^2}\,du \int_C F(iv)e^{-(\psi_0/2)v^2} \\ e^{-\psi_\tau uv} g_s(u, v, \tau)\,dv\end{aligned} \qquad (4.8\text{–}7)$$

where $g_s(u, v, \tau)$ is the ch. f. of $V_s(t)$ and $V_s(t+\tau)$ given by (4.8–5). This is a definite expression for $\Psi(\tau)$. All that follows is devoted to the evaluation of this integral and to the evaluation of

$$W(f) = 4\int_0^\infty \Psi(\tau)\cos 2\pi f\tau\,d\tau \qquad (4.6\text{–}1)$$

for the power spectrum of I.

Quite often $I(t)$ will contain *dc* and periodic components. It seems convenient to deal with these separately since they correspond to terms in $\Psi(\tau)$ which cause the integral (4.6–1) for $W(f)$ to diverge. In fact, from Section 2.2 it follows that a correlation function of the form

$$A^2 + \frac{C^2}{2}\cos 2\pi f_0\tau \qquad (2.2\text{–}3)$$

corresponds to a current

$$A + C\cos(2\pi f_0 t - \varphi) \qquad (2.2\text{–}2)$$

where the phase angle φ cannot be determined from (2.2–3) since it does not affect the average power.

Consider the correlation function for $V(t) = V_s(t) + V_N(t)$ given by (4.8–2). It is

$$\underset{T\to\infty}{\text{Limit}}\ \frac{1}{T}\left[\int_0^T V_s(t)V_s(t+\tau)\,dt + \int_0^T V_s(t)V_N(t+\tau)\,dt + \int_0^T V_N(t)V_s(t+\tau)\,dt + \int_0^T V_N(t)V_N(t+\tau)\,dt\right] \tag{4.8–8}$$

Since $V_s(t)$ and $V_N(t)$ are unrelated the contributions of the second and third integrals vanish leaving us with the result

$$\text{Correlation function of } V(t) = \text{Correlation function of } V_s(t) + \text{Correlation function of } V_N(t). \tag{4.8–9}$$

Now as $\tau \to \infty$ the correlation function of $V_N(t)$ becomes zero while that of $V_s(t)$ becomes of the type (2.2–3) given above. Hence the correlation function of the regular voltage $V_s(t)$ may be obtained from $V(t)$ by letting $\tau \to \infty$ and picking out the non-vanishing terms. Although we have been speaking of $V(t)$, the same results hold for $I(t)$ and this process may be used to pick out those parts of $\Psi(\tau)$ which correspond to the *dc* and periodic components of $I(t)$. Thus, if we look at (4.8–7) we see that as $\tau \to \infty$, $\psi_\tau \to 0$, while the $g_s(u, v, \tau)$ corresponding to $V_s(t)$ given by (4.8–5) remains unchanged in general magnitude. This last statement may be hard to see, but examination of the cases discussed later show that it is true, at least for these cases. Thus the portion of $\Psi(\tau)$ corresponding to the *dc* and periodic components of $I(t)$ is, setting $\psi_\tau = 0$ in (4.8–7),

$$\Psi_\infty(\tau) = \frac{1}{4\pi^2}\int_C F(iu)e^{-(\psi_0/2)u^2}\,du \int_C F(iv)e^{-(\psi_0/2)v^2} g_s(u, v, \tau)\,dv \tag{4.8–10}$$

where the subscript ∞ indicates that $\Psi_\infty(\tau)$ is that part of $\Psi(\tau)$ which does not vanish as $\tau \to \infty$.

We may write (4.8–9), when applied to $I(t)$, as

$$\Psi(\tau) = \Psi_\infty(\tau) + \Psi_c(\tau) \tag{4.8–11}$$

where $\Psi_c(\tau)$ is the correlation function of the "continuous" portion of the power spectrum of $I(t)$.

Incidentally, the separation of $\Psi(\tau)$ into the two parts shown in (4.8–11) may be avoided if one is willing to use the $\delta(f)$ functions in order to interpret the integral in (4.6–1) as explained in Section 2.2. This method gives the proper *dc* and sinusoidal components even though (4.6–1) does not converge (because of the presence of the terms leading to $\Psi_\infty(\tau)$).

4.9 Noise Plus Sine Wave Applied to Non-Linear Device

In order to illustrate the characteristic function method described in Section 4.8 we shall consider the case of a non-linear device specified by

$$I = \frac{1}{2\pi}\int_C F(iu)e^{iVu}\,du \tag{4A-1}$$

when V consists of a noise voltage plus a sine wave:

$$V(t) = P\cos pt + V_N(t) \tag{4.1-13}$$

As usual, $V_N(t)$ has the power spectrum $w(f)$ and the correlation function $\psi(\tau)$. $\psi(\tau)$ is often written as ψ_τ for the sake of shortness. Comparing (4.1-13) with (4.8-2) gives

$$V_s(t) = P\cos pt \tag{4.9-1}$$

Our first task is to compute the ch. f. $g_s(u, v, \tau)$ for the pair of random variables $V_s(t)$ and $V_s(t+\tau)$. We do this by using the integral (4.8-5):

$$\begin{aligned} g_s(u, v, \tau) &= \operatorname*{Limit}_{T\to\infty}\frac{1}{T}\int_0^T \exp\left[iuP\cos pt + ivP\cos p(t+\tau)\right]dt \\ &= J_0(P\sqrt{u^2+v^2+2uv\cos p\tau}) \end{aligned} \tag{4.9-2}$$

where J_0 is a Bessel function. The integration is performed by writing

$$\begin{aligned} u\cos pt + v\cos p\,(t+\tau) &= (u + v\cos p\tau)\cos pt - v\sin p\tau\sin pt \\ &= \sqrt{u^2+v^2+2uv\cos p\tau}\cos(pt + \text{phase angle}) \end{aligned}$$

and using the integral

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos t}\,dt$$

The correlation function for (4.1-13) has also been given in Section 3.10.

The correlation function $\Psi(\tau)$ for $I(t)$ may now be obtained by substituting the above expressions in (4.8-7)

$$\begin{aligned} \Psi(\tau) = \frac{1}{4\pi^2}\int_C du\,F(iu)e^{-(\psi_0/2)u^2}\int_C dv\,F(iv)e^{-(\psi_0/2)v^2} \\ e^{-\psi_\tau uv}J_0(P\sqrt{u^2+v^2+2uv\cos p\tau}). \end{aligned} \tag{4.9-3}$$

$\Psi_\infty(\tau)$, the correlation function for the d.c. and periodic components of I, may, according to (4.8-10), be obtained from this by setting ψ_τ equal to zero.

When we have a particular non-linear device in mind the appropriate $F(iu)$ may often be obtained from Appendix 4A. For example, $F(iu)$ for a linear rectifier is $-u^{-2}$. Inserting this value in (4.9-3) gives a definite

double integral for $\Psi(\tau)$. If there were some easy way to evaluate this integral then everything would be fine. Unfortunately, no simple method of evaluation has yet been found. However, one method is available which is closely related to the direct method used by Bennett. It is based on the expansion

$$g_s(u, v, \tau) = J_0(P\sqrt{u^2 + v^2 + 2uv \cos p\tau}) = \sum_{n=0}^{\infty} \epsilon_n (-)^n J_n(Pu) J_n(Pv) \cos np\tau \qquad (4.9\text{–}4)$$

$$\epsilon_0 = 1, \quad \epsilon_n = 2 \quad \text{for} \quad n \geq 1$$

This expansion enables us to write the troublesome terms in (4.9–3) as

$$e^{-\psi_\tau uv} J_0(P\sqrt{u^2 + v^2 + 2uv \cos p\tau}) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-)^{n+k} \epsilon_n \cos np\tau \frac{(\psi_\tau uv)^k}{k!} J_n(Pu) J_n(Pv) \qquad (4.9\text{–}5)$$

The virtue of this double sum is that it simplifies the integration. Thus, putting it in (4.9–3) and setting

$$h_{nk} = \frac{i^{n+k}}{2\pi} \int_C F(iu) u^k J_n(Pu) e^{-(\psi_0/2)u^2} du \qquad (4.9\text{–}6)$$

gives

$$\Psi(\tau) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \psi_\tau^k h_{nk}^2 \epsilon_n \cos np\tau \qquad (4.9\text{–}7)$$

The correlation function $\Psi_\infty(\tau)$ for the dc and periodic components of I are obtained by letting $\tau \to \infty$ where $\psi_\tau \to 0$. Only the terms for which $k = 0$ remain:

$$\Psi_\infty(\tau) = \sum_{n=0}^{\infty} \epsilon_n h_{n0}^2 \cos np\tau \qquad (4.9\text{–}8)$$

Comparing this with the known fact that the correlation function of

$$A + C \cos (2\pi f_0 t - \varphi) \qquad (2.2\text{–}2)$$

is

$$A^2 + \frac{C^2}{2} \cos 2\pi f_0 \tau \qquad (2.2\text{–}3)$$

and remembering that ϵ_0 is one while ϵ_n is two for $n \geq 1$ shows that

$$\begin{aligned} &\text{Amplitude of dc component of } I = h_{00} \\ &\text{Amplitude of } \frac{np}{2\pi} \text{ component of } I = 2h_{n0} \end{aligned} \qquad (4.9\text{–}9)$$

Incidentally, these expressions for the amplitudes follow almost at once from the direct method of solution. This will be shown in connection with equation (4.9–17).

Since the correlation function $\Psi_c(\tau)$ for the continuous portion $W_c(f)$ of the power spectrum for I is given by

$$\Psi_c(\tau) = \Psi(\tau) - \Psi_\infty(\tau), \tag{4.8–11}$$

we also have

$$\Psi_c(\tau) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k!} \psi_\tau^k h_{nk}^2 \epsilon_n \cos np\tau \tag{4.9–10}$$

When this is substituted in

$$W_c(f) = 4 \int_0^\infty \Psi_c(\tau) \cos 2\pi f\tau \, d\tau \tag{4.9–11}$$

we obtain

$$W_c(f) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{2\epsilon_n}{k!} h_{nk}^2 \left[G_k\left(f - \frac{np}{2\pi}\right) + G_k\left(f + \frac{np}{2\pi}\right)\right] \tag{4.9–12}$$

where

$$G_k(f) = \int_0^\infty \psi_\tau^k \cos 2\pi f\tau \, d\tau \tag{4.9–13}$$

is the function studied in Appendix 4C. $G_k(f)$ is an even function of f. The double series (4.9–12) for W_c looks rather formidable. However, when we are interested in a particular portion of the frequency spectrum often only a few terms of the series are needed.

It has been mentioned above that the direct method of obtaining the output power spectrum is closely related to the equations just derived. We now study this relation.

We start with the following result from modulation theory[50]: Let the voltage

$$\begin{aligned} V &= P_0 \cos x_0 + P_1 \cos x_1 + \cdots + P_N \cos x_N \\ x_k &= p_k t, \qquad k = 0, 1, \cdots N, \end{aligned} \tag{4.9–14}$$

where the p_k's are incommensurable, be applied to the device (4A–1). The output current is

$$I = \sum_{m_0=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \tfrac{1}{2} A_{m_0 \cdots m_N} \epsilon_{m_0} \cdots \epsilon_{m_N} \cos m_0 x_0 \cos m_1 x_1 \cdots \cos m_N x_N \tag{4.9–15}$$

[50] Bennett and Rice, "Note on Methods of Computing Modulation Products," *Phil. Mag.* S.7, V. 18, pp. 422–424, Sept. 1934, and Bennett's paper cited in Section 4.0.

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$. When the product of the cosines is expressed as a sum of cosines of the angles $m_0 x_0 \pm m_1 x_1 \cdots \pm m_N x_N$, it is seen that the coefficient of the typical term is $A_{m_0 \cdots m_N}$, except when all the m's are zero in which case it is $\frac{1}{2}A_{0\cdots 0}$. Thus

$$\begin{aligned} \tfrac{1}{2}A_{00\cdots 0} &= \text{dc component of } I \\ |A_{m_0 \cdots m_N}| &= \text{amplitude of component of frequency} \\ &\quad \frac{1}{2\pi}|m_0 p_0 \pm m_1 p_1 \pm \cdots \pm m_N p_N| \end{aligned} \tag{4.9–16}$$

For all values of the m's,

$$\begin{aligned} A_{m_0 \cdots m_N} &= \frac{i^M}{\pi}\int_C F(iu) \prod_{r=0}^{N} J_{m_r}(P_r u)\, du \\ M &= m_0 + m_1 + \cdots + m_N \end{aligned} \tag{4.9–17}$$

Following Bennett's procedure, we identify V as given by (4.9–14), with

$$V = P\cos pt + V_N \tag{4.1–13}$$

by setting $P_0 = P$, $p_0 = p$, and representing the noise voltage V_N by the sum of the remaining terms. Since this makes P_1, P_N all very small, Laplace's process indicates that in (4.9–17) we may put

$$\begin{aligned} \prod_{r=1}^{N} J_0(P_r u) &\approx \exp -\frac{u^2}{4}(P_1^2 + \cdots + P_N^2) \\ &\approx e^{-\psi_0 u^2/2} \end{aligned} \tag{4.9–18}$$

We have used the fact that ψ_0 is the mean square value of V_N. It follows from these equations that

$$\text{dc component of } I = \frac{1}{2\pi}\int_C F(iu)J_0(Pu)e^{(-\psi_0/2)u^2}\, du$$

$$\text{Component of frequency } \frac{np}{2\pi} = \frac{i^n}{\pi}\int_C F(iu)J_n(Pu)e^{-\psi_0 u^2/2}\, du$$

These results are identical with those of (4.9–9).

The equations just derived show that h_{n0} is to be associated with the n^{th} harmonic of p. In much the same way it may be shown that h_{nk} is to be associated with the modulation products arising from the n^{th} harmonic of p and k of the elementary sinusoidal components representing V_N. We consider only combinations of the form $p_1 \pm p_2 \pm p_3$, taking $k = 3$ for example, and neglect terms of the form $3p_1$ and $2p_1 \pm p_2$. The former type is much more numerous, there being about N^3 of them while there are only about N and N^2, respectively, of the latter type.

We again take $k = 3$ and consider m_1, m_2, m_3 to be one, and m_4, $\cdots$ m_N to be zero, corresponding to the modulation product $np \pm p_1 \pm p_2 \pm p_3$. By making the same sort of approximations as Bennett does we find

$$A_{n,1,1,1,0,0\cdots 0} = \frac{i^{n+3}}{\pi} \frac{P_1 P_2 P_3}{8} \int_C F(iu) J_n(Pu) u^3 e^{(-u^2/2)\psi_0} \, du$$

$$= \frac{P_1 P_2 P_3}{4} h_{n3}$$

When any other modulation product of the form $np \pm p_{r_1} \pm p_{r_2} \pm p_{r_3}$ is considered we get a similar expression in which $P_1P_2P_3$ is replaced by $P_{r_1}P_{r_2}P_{r_3}$. This may be done for any value of k. The result indicates that h_{nk}, and consequently also the $(n, k)^{\text{th}}$ terms in the double series (4.9–10) and (4.9–12) for $\Psi_c(\tau)$ and $W_c(f)$, are to be associated with the modulation products of order (n, k), the n referring to the signal and the k to the noise components.

We now may state a theorem due to Middleton regarding the total power in the modulation products of a given order. For a given non-linear device (i.e. $F(iu)$ is given), the total power which would be dissipated by all of the modulation products which are of order (n, k) if I were to flow through a resistance of one ohm is

$$\Psi_{nk}(0) = \frac{\epsilon_n [\psi(0)]^k}{k!} h_{nk}^2 = \frac{\epsilon_n [\overline{V_N^2}]^k h_{nk}^2}{k!} \qquad (4.9\text{–}19)$$

The important feature of this expression is that it depends only on the r.m.s. value of V_N and on $F(iu)$. It depends not at all upon the spectral distribution of the noise power in the input.

The proof of (4.9–19) is based on the relation

$$\Psi_{nk}(0) = \int_0^\infty W_{nk}(f) \, df$$

between the total power dissipated by all the (n, k) order products and the corresponding correlation function obtained from (4.9–7).

This theorem has been used by Middleton to show that when the input is confined to a relatively narrow frequency band, so that the output spectrum consists of bands, the power in each band depends only on $\overline{V_N^2}$ and not on the spectrum of V_N.

4.10 Miscellaneous Results Obtained by Correlation Function Method

In this section a number of results which may be obtained from the theory given in the sections following 4.6 are given.

When the input to the square law device

$$I = \alpha V^2 \tag{4.1-1}$$

consists of noise only, so that $V = V_N$, the correlation function for I is

$$\Psi(\tau) = \alpha^2[\psi_0^2 + 2\psi_\tau^2] \tag{4.10-1}$$

where ψ_τ is the correlation function of V_N. This may be compared with equation (3.9-7). When V is general,

$$\begin{aligned}\Psi(\tau) &= \text{ave. } I(t)I(t+\tau) \\ &= \text{ave. } \alpha^2 V^2(t)V^2(t+\tau) \\ &= \alpha^2 \times \text{Coefficient of } \frac{(iu)^2}{2!}\frac{(iv)^2}{2!} \text{ in power series expansion} \\ &\qquad \text{of ch. f. of } V(t),\ V(t+\tau)\end{aligned} \tag{4.10-2}$$

where we have used a known property of the characteristic function. An expression for the ch. f., denoted by $g(u, v, \tau)$, is given by (4.8-4). For example, when V consists of a sine wave plus noise, (4.1-13), the ch. f. is obtainable from (4.9-3). Hence,

$$\begin{aligned}\Psi(\tau) &= \text{Coeff. of } \frac{u^2v^2}{4} \text{ in expansion of} \\ &\qquad \alpha^2 J_0\left(P\sqrt{u^2 + v^2 + 2uv\cos p\tau}\right) \\ &\qquad \times \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \\ &= \alpha^2\left[\left(\frac{P}{2} + \psi_0\right)^2 + \frac{P^4}{8}\cos 2p\tau + 2P^2\psi_\tau \cos p\tau + 2\psi_\tau^2\right]\end{aligned} \tag{4.10-3}$$

The first two terms give the dc and second harmonic. The last two terms may be used to compute $W_c(f)$ as given by (4.5-13).

Expressions (4.10-1) and (4.10-3) are special cases of results obtained by Middleton who has studied the general theory of the quadratic rectifier by using the Van Vleck-North method, described in Section 4.7.

As an example to which the theory of Section 4.9 may be applied we consider the sine wave plus noise, (4.1-13), to be applied to the ν-law rectifier

$$\begin{aligned} I &= 0, \qquad V < 0 \\ I &= V^\nu, \qquad V > 0 \end{aligned} \tag{4.10-4}$$

From the table in Appendix 4A it is seen that

$$F(iu) = \Gamma(\nu + 1)(iu)^{-\nu-1}$$

and that the path of integration C runs along the real axis from $-\infty$ to ∞ with a downward indentation at the origin. The integral (4.9–6) for h_{nk} becomes

$$h_{nk} = \frac{i^{n+k-\nu-1}}{2\pi}\,\Gamma(\nu+1)\int_C u^{k-\nu-1}J_n(Pu)e^{-(\psi_0/2)u^2}\,du$$

$$= \frac{\left(\frac{\psi_0}{2}\right)^{(\nu-k)/2} x^{n/2}\,\Gamma(\nu+1)}{2\Gamma\left(\frac{2-k-n+\nu}{2}\right)n!}\,{}_1F_1\left(\frac{k+n-\nu}{2};\, n+1;\, -x\right) \qquad (4.10\text{–}5)$$

$$x = \frac{P^2}{2\psi_0}$$

where the integration has been performed by expanding $J_n(Pu)$ in powers of u and using

$$\int_C e^{-au^2}u^{2\lambda-1}\,du = ie^{-\lambda i\pi}a^{-\lambda}\sin\lambda\pi\Gamma(\lambda)$$

$$= \frac{a^{-\lambda}}{2}(1-e^{-2\lambda i\pi})\Gamma(\lambda) \qquad (4.10\text{–}6)$$

$$= \frac{i\pi e^{-\lambda i\pi}}{a^{\lambda}\Gamma(1-\lambda)}$$

it being understood that $\arg u = 0$ on the positive portion of C.

From (4.9–9), the dc component of I is

$$h_{00} = \frac{\Gamma(1+\nu)}{2\Gamma\left(1+\frac{\nu}{2}\right)}\left(\frac{\psi_0}{2}\right)^{\nu/2}{}_1F_1\left(-\frac{\nu}{2};\, 1;\, -x\right) \qquad (4.10\text{–}7)$$

which reduces to the expression (4.2–3) when $\nu = 1$ for the linear rectifier (aside from the factor α).

When the input (sine wave plus noise) is confined to a relatively narrow band, and when we are interested in the low frequency output, consideration of the modulation products suggests that we consider the difference products from the products of order (0, 0), (0, 2), (0, 4), $\cdots$ (1, 1), (1, 3), $\cdots$ (2, 0), (2, 2), $\cdots$ etc. where the typical product is of order (n, k). The orders (0, 0) and (2, 0) give the dc and second harmonic and hence are not considered in the computation of $W_c(f)$. Of the remaining terms, either (0, 2) or (1, 1) gives the greatest contribution to the series (4.9–12) and (4.9–10) for $W_c(f)$ and $\Psi_c(\tau)$. The remaining terms contribute less and less as n and

k increase. The low frequency portion of the continuous portion of the output power spectrum is then, from (4.9–12),

$$W_c(f) = \frac{4}{2!} h_{02}^2 G_2(f) + \frac{4}{4!} h_{04}^2 G_4(f) + \cdots$$
$$+ \frac{4}{1!} h_{11}^2 [G_1(f - f_0) + G_1(f + f_0)] + \frac{4}{3!} h_{13}^2 [G_3(f - f_0) \quad (4.10\text{–}8)$$
$$+ G_3(f + f_0)] + \frac{4}{2!} h_{22}^2 [G_2(f - 2f_0) + G_2(f + 2f_0)] + \cdots$$

From Table 2 of Appendix 4C we may pick out the low frequency portions of the G's. It must be remembered that $G_m(x)$ is an even function of x and that $0 < f \ll f_0$.

As an example we take the input noise V_N to have the same $w(f)$ and $\psi(\tau)$ as Filter a, the normal law filter, of Appendix 4C, so that

$$w(f) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(f-f_0)^2/2\sigma^2}$$

and assume that the sine wave signal is at the middle of the band, giving $p = 2\pi f_0$. Thus, from (4.10–8), for low frequencies and the normal law distribution of the input noise power,

$$W_c(f) = \frac{1}{4\sigma\sqrt{\pi}} h_{02}^2 \psi_0^2 e^{-f^2/4\sigma^2} + \frac{1}{64\sigma\sqrt{2\pi}} h_{04}^2 \psi_0^4 e^{-f^2/8\sigma^2}$$
$$+ \frac{2}{\sigma\sqrt{2\pi}} h_{11}^2 \psi_0 e^{-f^2/2\sigma^2} + \frac{1}{4\sigma\sqrt{6\pi}} h_{13}^2 \psi_0^3 e^{-f^2/6\sigma^2} \quad (4.10\text{–}9)$$
$$+ \frac{1}{4\sigma\sqrt{\pi}} h_{22}^2 \psi_0^2 e^{-f^2/4\sigma^2} + \cdots$$

Although we have been speaking of the ν-law rectifier, equation (4.10–9) gives the low frequency portion of $W_c(f)$, corresponding to a normal law noise power, for any non-linear device provided the proper h_{nk}'s are inserted.

When we set ν equal to one in the expression (4.10–5) for h_{nk} we may obtain the results given by Bennett. Middleton has studied the output of a biased linear rectifier, when the input consists of a sine wave plus noise, and also the special case of the unbiased linear rectifier. He has computed the output for a wide range of the ratios P^2/ψ_0, B^2/ψ_0 where B is the bias. In order to cover the entire range he had to derive two series for the corresponding h_{nk}'s, each series being suitable for its particular portion of the range.

A special case of (4.10–9) occurs when noise alone is applied to a linear rectifier. The low frequency portion of the output power spectrum is

$$W_c(f) = \frac{\psi_0}{\pi} \sum_{m=1}^{\infty} \frac{(-\tfrac{1}{2})_m(-\tfrac{1}{2})_m}{m!\,m!} \frac{1}{\sigma\sqrt{4m\pi}} e^{-f^2/4m\sigma^2}$$

$$= \frac{\psi_0 \pi^{-3/2}}{2\sigma}\left[\tfrac{1}{4}e^{-f^2/4\sigma^2} + \frac{1}{64\sqrt{2}} e^{-f^2/8\sigma^2} + \frac{1}{256\sqrt{3}} e^{-f^2/12\sigma^2} + \cdots\right] \tag{4.10–10}$$

where we have used (4.7–6) and Table 2 of Appendix 4C.

The correlation function of

$$V_s = P \cos pt + Q \cos qt,$$

where p and q are incommensurable, is

$$J_0(P\sqrt{u^2 + v^2 + 2uv \cos p\tau}) \times J_0(Q\sqrt{u^2 + v^2 + 2uv \cos q\tau})$$

From equations (4.9–16) and (4.9–17) it is seen immediately that

$$h_{000} = \frac{1}{2\pi}\int_C F(iu)J_0(Pu)J_0(Qu)e^{-(u^2/2)\psi_0}\,du \tag{4.10–11}$$

is the d.c. component of I when the applied voltage is

$$P \cos pt + Q \cos qt + V_N\,. \tag{4.1–4}$$

J. R. Ragazzini has obtained an approximate expression for the output power spectrum when the voltage

$$\begin{aligned} V &= V_s + V_N \\ V_s &= Q(1 + r \cos pt)\cos qt \end{aligned} \tag{4.10–12}$$

is impressed on a linear rectifier.[46] In terms of our notation his expression for the continuous portion of the power spectrum is (for low frequencies)

$$W_c(f) = \frac{1}{\pi^2\alpha^2(Q^2 + 2\psi_0)} \times \left[\begin{array}{l} W_c(f) \text{ given by equation} \\ (4.5\text{–}17) \text{ for square law device} \end{array}\right] \tag{4.10–13}$$

The α^2 is put in the denominator to cancel the α^2 in the expression (4.5–17). We take the linear rectifier to be

$$I = \begin{cases} 0, & V < 0 \\ V, & 0 < V \end{cases} \tag{4.10–14}$$

and replace the index of modulation, k, in (4.5–17) by r.

[46] Equation (12), "The Effect of Fluctuation Voltages on the Linear Detector," *Proc. I.R.E.*, V. 30, pp. 277–288 (June 1942).

Ragazzini's formula is quite accurate when the index of modulation r is small, especially when $y = Q^2/(2\psi_0)$ is large. To show this we put $r = 0$ in (4.10–13) and obtain

$$W_c(f) = \frac{1}{\pi^2(Q^2 + 2\psi_0)}\left[Q^2 w(f_q - f) + Q^2 w(f_q + f) + \int_{-\infty}^{+\infty} w(x)w(f - x)\,dx\right] \tag{4.10–15}$$

where $f_q = q/(2\pi)$. This is to be compared with the low frequency portion of $W_c(f)$ obtained by specializing (4.10–8) to obtain the output power spectrum of a linear rectifier when the input consists of a sine wave plus noise. The leading terms in (4.10–8) give

$$W_c(f) = h_{11}^2[w(f_q - f) + w(f_q + f)] + h_{02}^2 \frac{1}{4}\int_{-\infty}^{+\infty} w(x)w(f - x)\,dx \tag{4.10–16}$$

The values of the h's appropriate to a linear rectifier are obtained by setting $\nu = 1$ in (4.10–5) and noticing that Q now plays the role of P.

$$\begin{aligned} h_{11} &= \frac{1}{2}\left(\frac{y}{\pi}\right)^{1/2} {}_1F_1(\tfrac{1}{2}; 2; -y) \\ h_{02} &= (2\pi\psi_0)^{-1/2} {}_1F_1(\tfrac{1}{2}; 1; -y) \\ y &= Q^2/(2\psi_0) \end{aligned} \tag{4.10–17}$$

Incidentally, the first approximation to the output of a linear rectifier given by (4.10–16) is interesting in its own right. Fig. 9 shows the low frequency portion of $W_c(f)$ as computed from (4.10–16) when the input noise is uniformly distributed over a narrow frequency band of width β, f_q being the mid-band frequency. h_{11} and h_{02} may be obtained from the curves shown in Fig. 10. In these figures P and x replace Q and y of (4.10–17) in order to keep the notation the same as in Fig. 8 for the square law device. These curves may also be obtained from equations (33) to (43) of Bennett's paper.

The following values are useful for our comparison.

When $x = 0$	When x is large
$h_{11} = 0$	$h_{11} = 1/\pi$
$h_{02} = (2\pi\psi_0)^{-1/2}$	$h_{02} = 1/(\pi Q)$.

(4.10–18)

The values for large x are obtained from the asymptotic expansion (4B — 3) given in Appendix 4B.

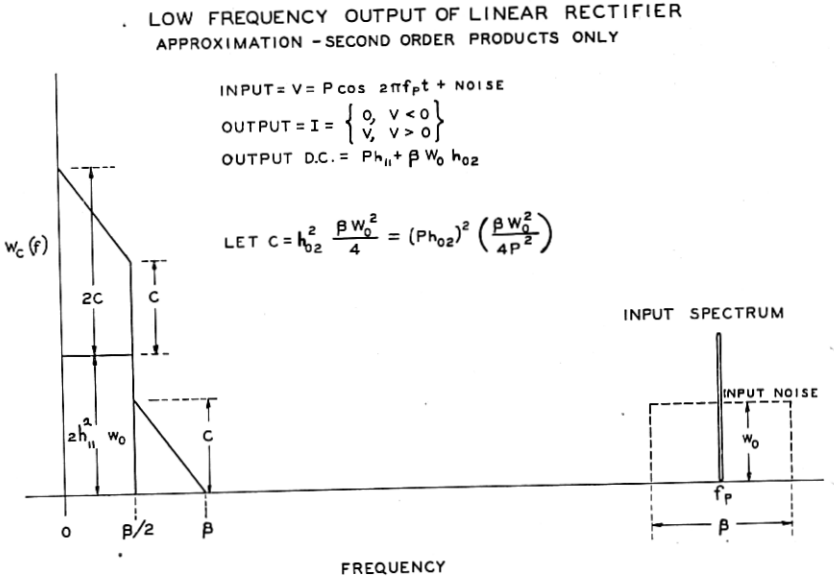

Fig. 9

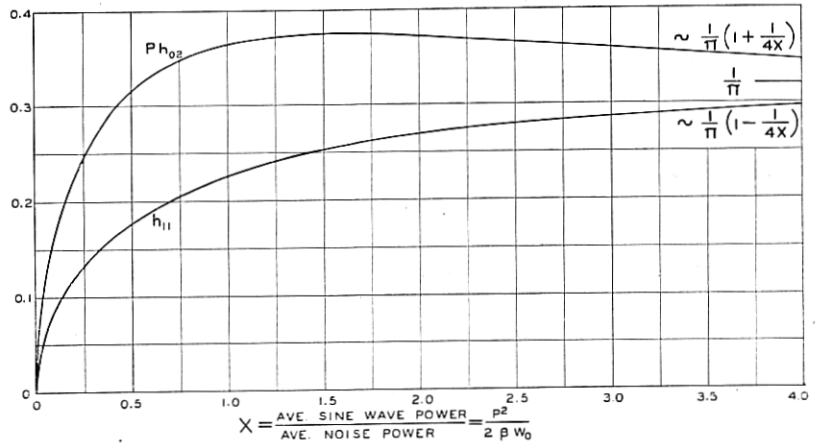

Fig. 10—Coefficients for linear detector output shown on Fig. 9

$$Ph_{02} = \sqrt{\frac{x}{\pi}}\,{}_1F_1(\tfrac{1}{2};\,1;\,-x) \qquad h_{11} = \frac{1}{2}\sqrt{\frac{x}{\pi}}\,{}_1F_1(\tfrac{1}{2};\,2;\,-x)$$

We make the first comparison between (4.10–15) and (4.10–16) by letting $Q \to \infty$. It is seen that both reduce to

$$W_c(f) = \frac{1}{\pi^2}\,[w(f_q - f) + w(f_q + f)] \qquad (4.10\text{–}19)$$

which shows that the agreement is perfect in this case. Next we let $Q = 0$. The two expressions then give

$$W_c(f) = \frac{1}{A2\pi\psi_0}\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx$$

where $A = \pi$ for Ragazzini's formula and $A = 4$ for (4.10–16). Thus the agreement is still quite good. The limiting value for (4.10–16) may also be obtained from (4.7–8).

Even if the index of modulation r is not negligibly small it may be shown that when $Q \to \infty$ $W_c(f)$ still approaches the value given by (4.10–19). Ragazzini's formula gives a somewhat larger answer because it includes the additional terms, shown in (4.5–17), which contain $k^2/4$, but this difference does not appear to be serious. If the $Q^2 + 2\psi_0$ in the denominator of (4.10–13) be replaced by $Q^2 + \frac{1}{2}Q^2k^2 + 2\psi_0$ the agreement is improved.

APPENDIX 4A

Table of Non-linear Devices Specified by Integrals

Quite a number of non-linear devices may be specified by integrals of the form

$$I = \frac{1}{2\pi}\int_C F(iu)e^{iVu}\,du \tag{4A–1}$$

where the function $F(iu)$ and the path of integration C are chosen to fit the device.* The table gives examples of such devices. Some important cases cannot be simply represented in this form. An example is the limiter

$$\begin{aligned} I &= -\alpha D, && V < -D \\ I &= \alpha V, && -D < V < D \\ I &= \alpha D, && D < V \end{aligned} \tag{4A–2}$$

which may be represented as

$$\begin{aligned} I &= \frac{2\alpha}{\pi}\int_0^{\infty} \sin Vu \sin Du \frac{du}{u^2} \\ &= -\alpha D + \frac{2\alpha}{2\pi i}\int_C e^{iVu} \sin Du \frac{du}{u^2} \end{aligned} \tag{4A–3}$$

where C runs from $-\infty$ to $+\infty$ and is indented downward at the origin. This is not of the form assumed in the theory of Part IV. However it appears that it would not be difficult to extend the theory in the particular case of the limiter.

* Reference 50 cited in Section 4.9.

Non-Linear Devices Specified by Integrals

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu}\, du$$

I	$F(iu)$	C	Type of Device
$I = \alpha V^n$, n integer	$\frac{\alpha\, n!}{(iu)^{n+1}}$	Positive Loop around $u = 0$	nth power device
$I = \alpha(V - B)^n$, n integer	$\frac{\alpha\, n!}{(iu)^{n+1}} e^{-iuB}$	Positive Loop around $u = 0$	nth power device with bias
$I = 0, \quad V < 0$ $I = \alpha V, \quad 0 < V$	$\frac{\alpha}{(iu)^2} = -\frac{\alpha}{u^2}$	Real u axis from $-\infty$ to $+\infty$ with downward indentation at $u = 0$	Linear rectifier cut-off at $V = 0$
$I = 0, \quad V < B$ $I = \alpha(V - B)^\nu$, $V > B$ ν any positive number	$\frac{\alpha\Gamma(\nu + 1)}{(iu)^{\nu+1}} e^{-iuB}$	"	νth power rectifier with bias
$I = 0, \quad V < 0$ $I = \alpha V, \quad 0 < V < D$ $I = \alpha D, \quad D < V$	$\frac{\alpha(1 - e^{-iuD})}{(iu)^2}$	"	Linear rectifier plus limiter
$I_+ = 0, \quad V < 0$ $I_- = \varphi(V), \quad V > 0$	$F(p) = \int_0^\infty e^{-pt}\varphi(t)\, dt$	"	

APPENDIX 4B

The Function ${}_1F_1(a; c; x)$

In problems concerning a sine wave plus noise the hypergeometric function

$$ {}_1F_1(a; c; z) = 1 + \frac{az}{c1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \qquad (4B\text{–}1)$$

arises. Here we state some of its properties which are of use in the theory of Part IV. Curves of ${}_1F_1(a; c; z)$ are given for $a = -4, -3.5 \cdots, 3.5, 4.0$ and $c = -1.5, -.5, +.5, 1, 1.5, 2, 3, 4$ in the 1938 edition, page 275, of "Tables of Functions", by Jahnke and Emde. A list of properties of the function and other references are also given. In addition to these references we mention E. T. Copson, "Functions of a Complex Variable" (Oxford, 1935), page 260.

If c is not a negative integer or zero

$$ {}_1F_1(a; c; z) = e^z\, {}_1F_1(c - a; c; -z). \qquad (4B\text{–}2)$$

When $R(z) > 0$ we have the asymptotic expansions

$$
\begin{aligned}
{}_1F_1(a; c; z) &\sim \frac{\Gamma(c)e^z}{\Gamma(a)z^{c-a}}\left[1 + \frac{(1-a)(c-a)}{1!z}\right. \\
&\left. + \frac{(1-a)(2-a)(c-a)(c-a+1)}{2!z^2} + \cdots\right] \\
{}_1F_1(a; c; -z) &\sim \frac{\Gamma(c)}{\Gamma(c-a)z^a}\left[1 + \frac{a(1+a-c)}{1!z}\right. \\
&\left. + \frac{a(a+1)(1+a-c)(2+a-c)}{2!z^2} + \cdots\right]
\end{aligned}
\tag{4B-3}
$$

Many of the hypergeometric functions encountered may be expressed in terms of Bessel functions of the first kind for imaginary argument. The connection may be made by means of the relation[51]

$$
{}_1F_1\left(\nu + \frac{1}{2}; 2\nu + 1; z\right) = 2^{2\nu}\Gamma(\nu+1)z^{-\nu}e^{z/2}I_\nu\left(\frac{z}{2}\right) \tag{4B-4}
$$

together with the recurrence relations

	F_{a+}	F_{a-}	F_{c+}	F_{c-}	F
1.	a	$(a-c)$			$c - 2a - z$
2.	ac		$(c-a)z$		$-c(a+z)$
3.	a			$1-c$	$c - a - 1$
4.		$-c$	$-z$		c
5.		$a-c$		$c-1$	$1 - a - z$
6.			$(c-a)z$	$c(c-1)$	$c(1-c-z)$

For example, the first recurrence relation is obtained from line 1 as follows

$$
aF(a+1; c; z) + (a-c)F(a-1; c; z) + (c - 2a - z)F(a; c; z) = 0 \tag{4B-5}
$$

These six relations between the contiguous ${}_1F_1$ functions are analogous to the 15 relations, given by Gauss, between the contiguous ${}_2F_1$ hypergeometric functions and may be derived from these by using

$$
{}_1F_1(a; c; z) = \operatorname*{Limit}_{b\to\infty} {}_2F_1\left(a, b; c; \frac{z}{b}\right) \tag{4B-6}
$$

A recurrence relation involving two ${}_1F_1$'s of the type (4B–4) may be obtained by replacing a by $a+1$ in the relation given by row four of the table

[51] G. N. Watson, "Theory of Bessel Functions" (Cambridge, 1922), p. 191.

and then eliminating ${}_1F_1(a+1;c;z)$ from this relation and the one obtained from row 3 of the table. There results

$$ {}_1F_1(a;c;z) = {}_1F_1(a;c-1;z) + \frac{za}{c(1-c)} F(a+1;c+1;z) \qquad (4B\text{–}7) $$

Setting ν equal to zero and one in (4B–4) and a equal to $\frac{1}{2}$, c equal to 2 in (4B–7) gives

$$ \begin{aligned} {}_1F_1\left(\frac{1}{2};1;z\right) &= e^{z/2} I_0\left(\frac{z}{2}\right) \\ {}_1F_1\left(\frac{3}{2};3;z\right) &= 4z^{-1}e^{z/2} I_1\left(\frac{z}{2}\right) \\ {}_1F_1\left(\frac{1}{2};2;z\right) &= e^{z/2}\left[I_0\left(\frac{z}{2}\right) - I_1\left(\frac{z}{2}\right)\right] \end{aligned} \qquad (4B\text{–}8) $$

Starting with these relations the relations in the table enable us to find an expression for ${}_1F_1(n+\frac{1}{2};m;z)$ where n and m are integers. A number of these are given in Bennett's paper. In particular, using (4B–2),

$$ {}_1F_1\left(-\frac{1}{2};1;-z\right) = e^{-z/2}\left[(1+z)I_0\left(\frac{z}{2}\right) + zI_1\left(\frac{z}{2}\right)\right]. \qquad (4B\text{–}9) $$

APPENDIX 4C

The Power Spectrum Corresponding to ψ_τ^n

Quite often we encounter the integral

$$ G_n(f) = \int_0^\infty [\psi(\tau)]^n \cos 2\pi f\tau \, d\tau \qquad (4C\text{–}1) $$

where $\psi(\tau)$ is the correlation function corresponding to the power spectrum $w(f)$. From the fundamental relation between $w(f)$ and $\psi(\tau)$ given by (2.1–5),

$$ G_1(f) = \frac{w(f)}{4} \qquad (4C\text{–}2) $$

The expression for the spectrum of the product of two functions enables us to write $G_n(f)$ in terms of $w(f)$. We shall use the following form of this expression: Let $F_r(f)$ be the spectrum of the function $\varphi_r(\tau)$ so that

$$ \varphi_r(\tau) = \int_{-\infty}^{+\infty} F_r(f) e^{2\pi i f\tau} \, df, \quad r = 1, 2 $$

$$ F_r(f) = \int_{-\infty}^{+\infty} \varphi_r(\tau) e^{-2\pi i f\tau} \, dt $$

Then

$$\int_{-\infty}^{+\infty} \varphi_1(\tau)\varphi_2(\tau)e^{-2\pi i f\tau}\,d\tau = \int_{-\infty}^{+\infty} F_1(x)F_2(f-x)\,dx \qquad (4C\text{–}3)$$

i.e., the spectrum of the product $\varphi_1(\tau)\varphi_2(\tau)$ is the integral on the right. If $\varphi_1(\tau)$ and $\varphi_2(\tau)$ are real even functions of τ, (4C–3) may be written as

$$\int_0^{\infty} \varphi_1(\tau)\varphi_2(\tau)\cos 2\pi f\tau\,d\tau = \frac{1}{2}\int_{-\infty}^{+\infty} F_1(x)F_2(f-x)\,dx \qquad (4C\text{–}4)$$

In order to obtain $G_2(f)$ we set $\varphi_1(\tau)$ and $\varphi_2(\tau)$ equal to $\psi(\tau)$. We may then use (4C-4) since $\psi(\tau)$ is an even real function of τ. When $\varphi_r(\tau)$ is an even real function of τ we see, from the Fourier integral for $F_r(f)$, that $F_r(f)$ must be an even real function of f. We therefore set

$$2F_r(f) = w(f), \qquad r = 1, 2$$

and define $w(f)$ for negative f by

$$w(-f) = w(f) \qquad (4C\text{–}5)$$

Equation (4C–4) then gives

$$\begin{aligned} G_2(f) &= \frac{1}{8}\int_{-\infty}^{+\infty} w(x)w(f-x)\,dx \\ &= \frac{1}{8}\int_0^{f} w(x)w(f-x)\,dx \\ &\quad + \frac{1}{4}\int_0^{\infty} w(x)w(f+x)\,dx \end{aligned} \qquad (4C\text{–}6)$$

where in the second equation only positive values of the argument of $w(f)$ appear.

In order to get $G_3(f)$ we set $\varphi_1(\tau)$ equal to $\psi(\tau)$, $2F_1(f)$ equal to $w(f)$, and $\varphi_2(\tau)$ equal to $\psi^2(\tau)$. Then

$$\begin{aligned} F_2(f) &= 2\int_0^{\infty} \varphi_2(\tau)\cos 2\pi f\tau\,d\tau \\ &= 2G_2(f) \end{aligned}$$

and from (4C–4) we obtain

$$\begin{aligned} G_3(f) &= \frac{1}{2}\int_{-\infty}^{+\infty} w(x)G_2(f-x)\,dx \\ &= \frac{1}{16}\int_{-\infty}^{+\infty} w(x)\,dx \int_{-\infty}^{+\infty} w(y)w(f-y)\,dy \end{aligned} \qquad (4C\text{–}7)$$

Equation (4C–7) suggests that we may write the expression for $G_2(f)$ as

$$G_2(f) = \frac{1}{2}\int_{-\infty}^{+\infty} w(x)G_1(f - x)\,dx \tag{4C–8}$$

This is seen to be true from (4C–2) and (4C–6). In fact it appears that

$$G_n(f) = \frac{1}{2}\int_{-\infty}^{+\infty} w(f - x)G_{n-1}(x)\,dx \tag{4C–9}$$

might be used for a step by step computation of $G_n(f)$.

We now consider $G_n(f)$ for the case of relatively narrow band pass filters. As examples we take filters whose characteristics give the following $w(f)$'s and $\psi(\tau)$'s

TABLE 1

Filter	$w(f)$ for $f > 0$	$\psi(\tau)$
a	$\frac{\psi_0}{\sigma\sqrt{2\pi}} e^{-(f-f_0)^2/2\sigma^2}$	$\psi_0 e^{-2(\pi\sigma\tau)^2} \cos 2\pi f_0\tau$
b	$\frac{\psi_0\,\alpha}{\pi} \frac{1}{\alpha^2 + (f - f_0)^2}$	$\psi_0 e^{-2\pi\alpha\lvert\tau\rvert} \cos 2\pi f_0\tau$
c	$w(f) = w_0 = \psi_0/\beta$ for $f_0 - \frac{\beta}{2} < f < f_0 + \frac{\beta}{2}$; $w(f) = 0$ elsewhere	$\psi_0 \frac{\sin \pi\beta\tau}{\pi\beta\tau} \cos 2\pi f_0\tau$

We shall refer to these filters as Filter a, Filter b, and Filter c, respectively. All have f_0 as the mid-frequency of the pass band. The constants have been chosen so that they all pass the same average power when a wide band voltage is applied:

$$\psi_0 = \int_0^\infty w(f)\,df = \text{mean square value of } I(t) \text{ or } V(t)$$

and it is assumed that $f_0 \gg \sigma$, $f_0 \gg \alpha$, $f_0 \gg \beta$ so that the pass bands are relatively narrow.

Expressions for $G_n(f)$ corresponding to several values of n are given in Table 2. When $n = 1$, $G_1(f)$ is simply $w(f)/4$. $G_2(f)$ is obtained by setting $n = 2$ in the definition (4C–1) for $G_n(f)$, squaring the $\psi(\tau)$'s of Table 1, and using

$$\cos^2 2\pi f_0\tau = \tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi f_0\tau$$

TABLE 2

$G_n(f)$	Filter a	Filter b
$G_1(f)$	$\frac{\psi_0}{4\sigma\sqrt{2\pi}}\, e^{-(f-f_0)^2/2\sigma^2}$	$\frac{\alpha\psi_0}{4\pi}\,\frac{1}{\alpha^2+(f-f_0)^2}$
$G_2(f)$	$\frac{\psi_0^2}{8\sigma\sqrt{4\pi}}\,[2e^{-f^2/4\sigma^2}+e^{-(f-2f_0)^2/4\sigma^2}]$	$\frac{2\alpha\psi_0^2}{8\pi}\left[\frac{2}{4\alpha^2+f^2}+\frac{1}{4\alpha^2+(f-2f_0)^2}\right]$
$G_3(f)$	$\frac{\psi_0^3}{16\sigma\sqrt{6\pi}}\,[3e^{-(f-f_0)^2/6\sigma^2}+e^{-(f-3f_0)^2/6\sigma^2}]$	$\frac{3\alpha\psi_0^3}{16\pi}\left[\frac{3}{9\alpha^2+(f-f_0)^2}+\frac{1}{9\alpha^2+(f-3f_0)^2}\right]$
$G_4(f)$	$\frac{\psi_0^4}{32\sigma\sqrt{8\pi}}\,[6e^{-f^2/8\sigma^2}+4e^{-(f-2f_0)^2/8\sigma^2}+e^{-(f-4f_0)^2/8\sigma^2}]$	$\frac{4\alpha\psi_0^4}{32\pi}\left[\frac{6}{16\alpha^2+f^2}+\frac{4}{16\alpha^2+(f-2f_0)^2}+\frac{1}{16\alpha^2+(f-4f_0)^2}\right]$
$G_n(f)$ n odd f small	0	0
$G_n(f)$ n even f small	$\frac{\psi_0^n\, n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}\,\frac{e^{-f^2/2n\sigma^2}}{2^{n+1}\,\sigma\sqrt{2n\pi}}$	$\frac{\psi_0^n\, n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!\,2^n}\,\frac{1}{2^{n+1}\,\pi n\alpha}\,\frac{1}{1+\left(\frac{f}{n\alpha}\right)^2}$
$G_n(f)$ n even n large f small	$\frac{1}{2\pi\sigma n}\, e^{-f^2/2n\sigma^2}$	$\frac{2}{\alpha(2\pi n)^{3/2}}\,\frac{1}{1+\left(\frac{f}{n\alpha}\right)^2}$
Filter c $G_1(f)$	$\frac{\psi_0}{4\beta}$ when $f_0-\frac{\beta}{2}<f<f_0+\frac{\beta}{2}$; 0 elsewhere	
Filter c $G_2(f)$		$\frac{\psi_0^2}{4\beta}\left(1-\frac{f}{\beta}\right)$ when $0\le f\le\beta$ $\frac{\psi_0^2}{8\beta^2}\,(f-2f_0+\beta)$ " $2f_0-\beta\le f\le 2f_0$ $\frac{\psi_0^2}{8\beta^2}\,(2f_0+\beta-f)$ " $2f_0\le f\le 2f_0+\beta$

The expression for $G_2(f)$ given in Table 2 corresponding to Filter c is exact. The expressions for Filters a and b give good approximations around $f = 0$ and $f = 2f_0$ where $G_2(f)$ is large. However, they are not exact because terms involving $f + 2f_0$ have been omitted. It is seen that all three G_2's behave in the same manner. Each has a peak symmetrical about $2f_0$ whose width is twice that of the original $w(f)$, is almost zero between 0 and $2f_0$, and rises to a peak at 0 whose height is twice that at $2f_0$.

$G_3(f)$ is obtained by cubing the $\psi(\tau)$ given in Table 1 and using

$$\cos^3 2\pi f_0\tau = \tfrac{3}{4}\cos 2\pi f_0\tau + \tfrac{1}{4}\cos 6\pi f_0\tau.$$

From the way in which the cosine terms combine with $\cos 2\pi f\tau$ in (4C–1) we see that $G_3(f)$, for our relatively narrow band pass filters, has peaks at f_0 and $3f_0$, the first peak being three times as high as the second. The expressions given for $G_3(f)$ and $G_4(f)$ are approximate in the same sense as are those for $G_2(f)$. It will be observed that the coefficients within the brackets, for Filters a and b, are the binomial coefficients for the value of n concerned. Thus for $n = 2$, they are 2 and 1, for $n = 3$ they are 3 and 1, and for $n = 4$ they are 6, 4, and 1.

The higher $G_n(f)$'s for Filters a and b may be computed in the same way. The integrals to be used are

$$\int_0^\infty e^{-2n(\pi\sigma\tau)^2}\cos 2\pi f\tau\, d\tau = \frac{e^{-f^2/2n\sigma^2}}{2\sigma\sqrt{2n\pi}}$$

$$\int_0^\infty e^{-2n\pi\alpha\tau}\cos 2\pi f\tau\, d\tau = \frac{1}{2\pi}\frac{n\alpha}{n^2\alpha^2 + f^2}$$

In many of our examples we are interested only in the values $G_n(f)$ for f near zero, i.e., only in that peak which is at zero. It is seen that $G_n(f)$ has such a peak only when n is even, this peak arising from the constant term in the expansion

$$\cos^{2k} x = \frac{1}{2^{2k-1}}\left[\cos 2kx + 2k\cos 2(k-1)x + \frac{(2k)(2k-1)}{2!}\cos 2(k-2)x + \cdots + \frac{(2k)!}{(k-1)!(k+1)!}\cos 2x + \frac{(2k)!}{k!k!2}\right]$$